

# 3-SPHERES IN THE 4-SPHERE AND PSEUDO-ISOTOPIES OF $S^1 \times S^3$

DAVID GABAI

ABSTRACT. We offer an approach to the smooth 4-dimensional Schoenflies conjecture via pseudo-isotopy theory.

## 0. INTRODUCTION

The 4-dimensional smooth Schoenflies conjecture (SS4) asserts that every embedded smooth 3-sphere in the 4-sphere bounds a smooth 4-ball. In 1958 Barry Mazur [Ma1] proved that such spheres bound topological 4-balls, as a special case of a more general result. A consequence of his elementary but strikingly original proof is the following.

**Theorem 0.1.** (*Mazur [Ma2], unpublished*) *If SS4 is false, then there exists a diffeomorphism  $\phi : S^1 \times S^3 \rightarrow S^1 \times S^3$  such that  $\phi$  is homotopic to id but  $\phi(x_0 \times S^3)$  is not isotopic to  $x_0 \times S^3$ , even after lifting to any finite sheeted covering of  $S^1 \times S^3$ .*

On the other hand it was recently proved in [BG] that there exists diffeomorphisms  $\phi \in \text{Diff}_0(S^1 \times S^3)$  such that  $\phi(x_0 \times S^3)$  is not isotopic to  $x_0 \times S^3$ . Here  $\text{Diff}_0(S^1 \times S^3)$  denotes the topological group of diffeomorphisms of  $S^1 \times S^3$  whose elements are homotopic to id, a group of index-8 in  $\text{Diff}(S^1 \times S^3)$ .

This paper introduces the study of  $\text{Diff}_0(S^1 \times S^3)$  through the use of pseudo-isotopy theory in an attempt to address the Schoenflies conjecture. Theorem 4.6 and Proposition 7.1 equate SS4 to a certain interpolation problem involving a regular homotopy whose finger and Whitney discs coincide near their boundaries. We use this to give a condition for showing that a Poincare 4-ball (i.e. a compact contractible 4-manifold with boundary  $S^3$ ) is a Schoenflies 4-ball (i.e. a closed complementary region of an embedded 3-sphere in  $S^4$ ).

Along the way we obtain the following.

**Theorem 0.2.** *If  $\phi \in \text{Diff}_0(S^1 \times S^3)$ , then  $\phi$  is stably isotopic to id.*

This means that for  $k$  sufficiently large, after isotoping  $\phi$  to be the id on a  $\sqcup_k B^4$ , and then extending to  $\hat{\phi} \in \text{Diff}_0(S^1 \times S^3 \#_k S^2 \times S^2)$  where the sum is taken inside the  $B^4$ 's and  $\hat{\phi}$  is id within the summands, then  $\hat{\phi}$  is isotopic to id.

**Remarks 0.3.** i) This is more generally true for diffeomorphisms of 4-manifolds pseudo-isotopic to id for which the Hatcher - Wagoner [HW] obstruction vanishes. See §3.

ii) In 1986 Frank Quinn [Qu] showed that a pseudo-isotopy  $f$  from id to  $\phi \in \text{Diff}_0(M)$ ,  $M$  a closed simply connected 4-manifold, is stably isotopic to id. Both Perron [Pe] and Quinn

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Version 0.624 June 10, 2022.

Partially supported by NSF grants DMS-1006553, DMS-1607374 and DMS-2003892.

*Primary class:* 57M99

*secondary class:* 57R52, 57R50, 57N50

*keywords:* 4-manifolds, Schoenflies, pseudo-isotopy.

[Qu] showed that such an  $f$  is topologically isotopic to  $\text{id}$ . This implies that  $\phi$  is stably isotopic to  $\text{id}$  and topologically isotopic to  $\text{id}$ .

We prove the following characterization of Schoenflies balls.

**Theorem 0.4.** *Every Schoenflies ball has a carving/surgery presentation.*

This means that it is obtained by a finite process starting with the 4-ball, attaching finitely many 2-handles, then carving finitely many 2-handles, then attaching finitely many 2-handles, etc., with every step happening in the 4-sphere. Actually we show that the presentation can be chosen to be of a special type called an *optimized  $F|W$ -carving/surgery presentation*. See Definitions 9.6 and 9.10 and Theorem 11.6.

**Underlying Conventions:** Unless said otherwise, this paper works in the smooth category and diffeomorphisms between oriented manifolds are orientation preserving.

**Acknowledgements 0.5.** We thank Valentin Poenaru for many conversations about mathematics over the last 30 years and for introducing me to Barry Mazur’s work. We thank Bob Edwards for long conversations over the last 15 years. We thank Toby Colding for discussions leading to the rediscovery of Mazur’s unpublished theorem. During 2016 part of this work, namely the reduction of the Schoenflies problem to a certain interpolation problem Theorem 6.5, was presented at the Bonn Max Planck Institute “Conference on 4-manifolds and knot concordance”, the “Benjamin Pierce Centennial Conference” at Harvard and at a meeting at Trinity College. In the interim have appeared the papers [Ga1], [ST], [BG] and [Gay] which have enabled a more concise exposition of that theorem. We thank Ryan Budney, David Gay and Hannah Schwartz for helpful conversations. This research was carried out in part during visits to Trinity College, Dublin, the Institute for Advanced Study and the IHES. We thank these institutions for their hospitality.

## 1. THE SCHOENFLIES CONJECTURE AND $\text{Diff}_0(S^1 \times S^3)$

Before proving Mazur’s Theorem 0.1 we record the following basic fact.

**Lemma 1.1.** *Let  $E, W \in S^4$  be distinct points. Two oriented 3-spheres  $\Sigma_0, \Sigma_1 \subset S^4 \setminus \{E, W\}$  are isotopic in  $S^4 \setminus \{E, W\}$  if and only if they are isotopic in  $S^4$  and they represent the same class in  $H_3(S^4 \setminus \{E, W\})$ .*  $\square$

*Proof of Theorem 0.1:* View  $S^4$  as  $[-\infty, \infty] \times S^3$  where  $-\infty \times S^3$  and  $\infty \times S^3$  are identified to points, which are respectively denoted  $W$  and  $E$ . If  $\Sigma \subset S^4$  is an oriented 3-sphere, then after isotopy, we can assume that  $\Sigma \subset (0, 1) \times S^3$  and homologous to  $0 \times S^3$  in  $(-\infty, \infty) \times S^3$ . The  $\mathbb{Z}$ -action  $t \times S^3 \rightarrow t+1 \times S^3$  induces a covering  $\pi : \mathbb{R} \times S^3 \rightarrow S^1 \times S^3$ . Let  $\Sigma' = \pi(\Sigma)$  and  $S'_0 = \pi(0 \times S^3)$ . By Theorem 3.13 [BG], there exists a diffeomorphism  $\phi \in \text{Diff}_0(S^1 \times S^3)$  such that  $\phi(S'_0) = \Sigma'$ . If after lifting to a finite cover  $\Sigma'$  is isotopic to  $S'_0$ , then  $\Sigma$  is isotopic to  $S_0$  in  $(-\infty, \infty) \times S^3$  and hence  $S^4$ . Thus if  $\Sigma$  is not isotopic to  $S_0$ , then  $\phi$  satisfies the conclusion of Theorem 0.1.  $\square$

**Remark 1.2.** On the other hand if  $\Sigma' \subset S^1 \times S^3$  lifts to  $\Sigma \subset \mathbb{R} \times S^3 \subset S^4$  which is isotopic to  $0 \times S^3$  viewed in  $S^4$ , then by Lemma 1.1 it would be isotopic to  $S'_0$  after lifting to a sufficiently high finite cover.

*Proof of Mazur's Schoenflies Theorem:* With notation as in the previous proof, let  $\mathcal{F}$  denote the  $S^3$ -fibration of  $S^1 \times S^3$  by pushing forward the standard fibration by  $\phi$ . This lifts to a  $S^3$ -fibration of  $\mathbb{R} \times S^3$  with  $\Sigma$  as a leaf. Thus, as in Mazur's original Schoenflies proof, the closure of each component of  $S^4 \setminus \Sigma$  is a smooth  $S^3 \times [0, \infty)$  whose end limits on the missing point.  $\square$

**Lemma 1.3.** *If  $\Sigma_0, \Sigma_1$  are disjoint, isotopic and non separating 3-spheres in  $S^1 \times S^3$ , then they bound a smooth product.*

*Proof.* This is immediate if  $\Sigma_0 = x_0 \times S^3$ . By [BG] we can assume this is the case.  $\square$

**Definition 1.4.** Call a compact oriented contractible 4-ball  $\Delta$  with  $\partial\Delta = S^3$  a *Schoenflies* 4-ball if  $\Delta$  embeds in  $S^3$ . A *Schoenflies* sphere is an oriented embedded 3-sphere in  $S^4$ .

The following is a restatement of a theorem of Bob Gompf.

**Theorem 1.5.** (Gompf [Go]) *Two Schoenflies balls  $\Delta_0, \Delta_1 \subset S^4$  are diffeomorphic if and only if they are ambiently isotopic.*

*Proof.* It suffices to consider the case  $\Delta_0 \cap \Delta_1 = \emptyset$ . Gompf shows that there exists a diffeomorphism  $\phi : (S^4, \Delta_0) \rightarrow (S^4, \Delta_1)$ . By precomposing  $\phi$  if necessary by a diffeomorphism supported away from  $\Delta_0 \cup \Delta_1$  we can assume that  $\phi$  is isotopic to id.  $\square$

**Notation 1.6.** If  $f \in \text{Diff}_0(S^1 \times S^3)$ , then  $\hat{f}$  will denote a lift to a finite sheeted covering space that will be determined by context and  $\tilde{f}$  will denote a lift to  $\mathbb{R} \times S^3$ . Unless said otherwise the particular lift is immaterial.

**Definition 1.7.** Two diffeomorphisms  $f, g \in \text{Diff}_0(S^1 \times S^3)$  are *S-equivalent* if there are lifts  $\hat{f}$  and  $\hat{g}$  of  $f$  and  $g$  to a finite sheeting covering space such that  $\hat{f}(Q)$  is isotopic to  $\hat{g}(Q)$  for some  $Q$  of the form  $t \times S^3$ .

Let  $f, g \in \text{Diff}_0(S^1 \times S^3)$ . We say that  $f$  *interpolates to*  $g \in \text{Diff}_0(\mathbb{R} \times S^3)$  if there exists a  $\tilde{h} \in \text{Diff}_0(\mathbb{R} \times S^3)$  such that  $\tilde{h}$  coincides with  $\tilde{f}$  (resp.  $\tilde{g}$ ) on the  $-\infty$  (resp.  $+\infty$ ) end of  $\mathbb{R} \times S^3$ .

**Proposition 1.8.** *Let  $f, g \in \text{Diff}_0(S^1 \times S^3)$ . The following are equivalent.*

- i)  $f$  and  $g$  are *S-equivalent*.
- ii) If  $P \subset S^1 \times S^3$  is a non separating 3-sphere, then  $\tilde{f}(\tilde{P})$  is isotopic to  $\tilde{g}(\tilde{P})$ . Here  $\tilde{P}$  is any lift of  $P$  to  $\mathbb{R} \times S^3$ .
- iii) If  $P \subset S^1 \times S^3$  is a non separating 3-sphere, then there exists a finite sheeted covering space  $\hat{V}$  such that  $\hat{f}(\hat{P})$  is isotopic to  $\hat{g}(\hat{P})$ . Here  $\hat{P}$  is any lift of  $P$  to  $\hat{V}$ .
- iv)  $f$  interpolates to  $g$  and  $g$  interpolates to  $f$  in  $\text{Diff}_0(\mathbb{R} \times S^3)$ .
- v) After passing to finite sheeted covering space  $\hat{f}$  is isotopic to  $\hat{g}$  modulo  $\text{Diff}_0(B^4 \text{ fix } \partial)$ .

*Proof.* i) implies v): Let  $Q = t \times S^3$ . Pass to a finite sheeted covering  $\hat{V}$  of  $S^1 \times S^3$  so that  $\hat{f}(\hat{Q})$  is isotopic to  $\hat{g}(\hat{Q})$ . By [BG]  $\hat{f}$  and  $\hat{g}$  are isotopic modulo  $\text{Diff}(B^4 \text{ fix } \partial)$ .  $\square$

v) implies iii): Pass to a finite sheeted covering so that v) holds. After isotopy  $\hat{f} = \hat{g}$  modulo  $\text{Diff}(B^4)$ . Since we can assume that the 4-ball is disjoint from  $\hat{P}$  the result follows.  $\square$

iii) if and only if ii): The only if direction is immediate and the other follows from the fact that an ambient isotopy of  $\tilde{f}(\tilde{P})$  to  $\tilde{g}(\tilde{P})$  in  $\mathbb{R} \times S^3$  can be taken to be compactly supported.  $\square$

iii) implies i): □

ii) implies iv): We show  $f$  interpolates to  $g$ . Consider  $\tilde{f}, \tilde{g} : \mathbb{R} \times S^3 \rightarrow \mathbb{R} \times S^3$ . Using Cerf [Ce2] after a compactly supported isotopy of  $\tilde{f}$  to  $\tilde{f}'$ , we can assume that  $\tilde{f}'|N(\tilde{Q}) = \tilde{g}|N(\tilde{Q})$ . Let  $L$  denote the closed complementary region lying to the  $-\infty$ -side of  $Q$  and  $R$  the component to the  $+\infty$  side. Define  $h : \mathbb{R} \times S^3 \rightarrow \mathbb{R} \times S^3$  by  $h|L = \tilde{f}'|L$  and  $h|R = \tilde{g}$ . A similar argument shows that  $g$  interpolates to  $f$ . □

iv) implies ii): Consider an interpolation  $\tilde{h}$  of  $\tilde{f}$  to  $\tilde{g}$ . Since  $\tilde{h}(t \times S^3)$  is isotopic to  $\tilde{h}(t' \times S^3)$  our assertion follows. □

**Corollary 1.9.** *Interpolation is an equivalence relation.* □

Any  $f \in \text{Diff}_0(S^1 \times S^3)$  is isotopic to one supported in  $S^1 \times B$ , for any 3-ball  $B \subset S^3$  and hence  $\text{Diff}_0(S^1 \times S^3)$  is abelian and composition is isotopic to the contenance of two such maps supported on disjoint  $S^1 \times B^3$ 's. E.g. see [BG]. From this the next result follows.

**Proposition 1.10.** *The groups of Schoenflies spheres, Schoenflies balls and S-equivalence classes of  $\text{Diff}_0(S^1 \times S^3)$  are naturally isomorphic. The isomorphism between Schoenflies balls and Schoenflies 3-spheres is induced by passing to the boundary, using the outward first orientation convention. The bijection between S-equivalence classes and spheres is given by  $f \rightarrow i \circ \tilde{f}(x_0 \times S^3)$ . Furthermore,*

*i) (law of composition) boundary connect sum for Schoenflies balls and composition for S-equivalence classes.*

*ii) (inverse) passing to complementary Schoenflies ball in  $S^4$  for Schoenflies balls, reversing orientation of Schoenflies spheres and inverse in  $\text{Diff}_0(S^1 \times S^3)$  for S-equivalence classes.*

**Notation 1.11.** If  $M$  is an oriented manifold, then  $\bar{M}$  denotes  $M$  oppositely oriented. If  $\Delta$  is a Schoenflies ball, then  $-\Delta$  denotes the complementary Schoenflies ball.

**Proposition 1.12.** *If  $\Delta^4, \Sigma^3, f$  correspond under Proposition 1.10, then  $\bar{\Delta}^4$  is diffeomorphic to  $r_{S^3}(\Delta^4)$  and corresponds to  $r_{S^3}(\Sigma^3)$  and  $r_{S^3}fr_{S^3} := \bar{f}$ . Here  $r_{S^3}$  denotes reflection in  $S^3$  either in  $\mathbb{R} \times S^3$  or  $S^1 \times S^3$ . Also  $r_{S^3}(\Delta^4)$  is oriented as a subspace of  $S^4$  and  $r_{S^3}(\Sigma^3)$  has the corresponding boundary orientation.* □

**Conjecture 1.13.** *If  $\Delta$  is a Schoenflies ball, then  $-\Delta = \bar{\Delta}$ . Equivalently, if  $\Delta$  is a Schoenflies ball, then  $\Delta \times I = B^5$ .*

The following characterization of the Schoenflies problem is well known.

**Definition 1.14.** If  $x \in \text{int}(M)$ , then let  $M_x$  denote  $M \setminus x$ . We say that a diffeomorphism  $\phi : M_x \rightarrow N_y$  induces the diffeomorphism  $\phi' : M \rightarrow N$  if there exists a compact 4-ball  $B \subset M$  such that  $x \in \text{int}(B)$  and  $\phi|M \setminus B = \phi'|M \setminus B$ .

**Theorem 1.15.** *The Schoenflies conjecture is true if and only if for every pair of compact 4-manifolds  $M, N$  a diffeomorphism  $\phi : M_x \rightarrow N_y$  induces a diffeomorphism  $\phi' : M \rightarrow N$ .* □

**Remark 1.16.** Deleting an interior point is not in general sufficient to make homeomorphic but non diffeomorphic manifolds diffeomorphic. Indeed, Akbulut's original cork [Ak] is an example of a compact contractible 4-manifold  $A$  with an involution  $f$  on  $\partial A$  such that  $f$  extends to a homeomorphism but not a diffeomorphism on  $A$ . This property continues to

hold even if  $A$  is punctured for Akbulut shows that there exists a curve  $\beta \subset \partial A$  that slices in  $A$  such that  $f(\beta)$  does not slice.

This leads to the following well known conjecture, the forward direction of which is immediate.

**Conjecture 1.17.** *The Poincare ball  $\Delta$  is a Schoenflies ball if and only if every knot  $K \subset \partial\Delta$  that slices in  $\Delta$ , slices in  $B^4$ .*

## 2. PSEUDO-ISOTOPY VS STABLE ISOTOPY

**Definition 2.1.** Let  $M$  be a compact oriented 4-manifold. A *pseudo-isotopy* from  $\phi_0$  to  $\phi_1$  is a diffeomorphism  $f : M \times I \rightarrow M \times I$  such that  $f|M \times 0 = \phi_0 \times 0$ ,  $f|\partial M \times I = \phi_0 \times \text{id}$  and  $f|M \times 1 = \phi_1 \times 1$ . In all cases in this paper,  $\phi_0|\partial M = \text{id}$ . Let  $M_k$  denote  $M \#_k S^2 \times S^2$ , where  $k \in \mathbb{N}$ . Here all the sums are taken in disjoint 4-balls  $B_1, \dots, B_k$ . We say  $\Phi : M_k \times [0, 1] \rightarrow M_k$  is a *stable isotopy* from  $\phi_0$  to  $\phi_1$  if for  $t = 0, 1$ ,  $\Phi_t|_{B_i \# S^2 \times S^2} = \text{id}$  and the induced maps  $\hat{\Phi}_t : M \rightarrow M$  which are id on each  $B_i$ , are isotopic to  $\phi_0$  and  $\phi_1$  respectively. When  $\phi_0 = \text{id}$  we say that  $\phi_1$  is the *map induced by the stable isotopy*. Let  $\text{Diff}_0^\Sigma(M)$  denote the subgroup of  $\text{Diff}_0(M)$  generated by elements stably isotopic to id.

**Lemma 2.2.** *If  $\Phi$  is a stable isotopy from  $\phi_0$  to  $\phi_1$ , then  $\Phi^{-1}$  defined by  $\Phi_t^{-1} = (\Phi_t)^{-1}$  is a stable isotopy from  $\phi_0^{-1}$  to  $\phi_1^{-1}$  and  $\bar{\Phi}$  defined by  $\bar{\Phi}_t = \Phi_{1-t}$  is a stable isotopy from  $\phi_1$  to  $\phi_0$ .  $\square$*

**Lemma 2.3.** *Pseudo-isotopy and stable isotopy are equivalence relations on  $\text{Diff}_0(M)$  where  $M$  is a compact oriented 4-manifold.  $\square$*

If  $M$  is a closed oriented simply connected 4-manifold and  $\phi \in \text{Diff}_0(M)$ , then by Kreck [Kr] p. 645 or [Qu]  $\phi$  is pseudo-isotopic to id. Quinn [Qu] used this to show that  $\phi$  is stably isotopic to id. In the topological category both Perron [Pe] and Quinn [Qu] showed that  $\phi$  is isotopic to id, so no stabilization is needed. Ruberman [Ru] constructed diffeomorphisms of closed simply connected 4-manifolds that are topologically but not smoothly isotopic to id.

Using the seminal work of Cerf [Ce3]; Hatcher, Wagoner and Igusa found three obstructions for a pseudo-isotopy of a compact manifold  $M$  to be isotopic to id when  $\dim(M) \geq 7$ . See [HW], [Ha2], [Ig1]. As detailed in [HW] various elements of the Hatcher-Wagoner theory function when  $\dim(M) \geq 4$ . The first obstruction,  $\Sigma(f) \in \text{Wh}_2(\pi_1(M))$ , is defined when  $\dim(M) \geq 4$  and is the exact obstruction to having a *nested eye* Cerf diagram with only critical points of index 2 and 3 and a gradient like vector field (glvf) that does not involve handle slides and has independent birth and death points. Chapter 5 §6 of [HW] shows that under these circumstances  $\Sigma(f) = 0$ . The converse follows from Proposition 3, P. 214 [HW] and its proof. Call such a nested eye 1-parameter family  $q_t$  together with its glvf  $v_t$  a *Hatcher - Wagoner family*.

We now briefly outline Quinn's proof that if  $M$  is a compact simply connected 4-manifold and  $\phi \in \text{Diff}_0(M)$  where  $\phi$  is pseudo-isotopic to id, then  $\phi$  is stably isotopic to id. Let  $f$  denote a pseudo-isotopy from id to  $\phi$ . Consider a 1-parameter family  $q_t : M \times I \rightarrow [0, 1]$ ,  $t \in [0, 1]$  so that  $q_0$  is the standard projection to  $[0, 1]$ ,  $q_1$  is the projection to  $[0, 1]$  induced from the pseudo-isotopy and  $q_t$  is a path from  $q_0$  to  $q_1$  in the space of smooth maps  $M \times I \rightarrow [0, 1]$  that agrees with  $q_0$  on  $N(\partial M \times I)$ . We can assume that this is a Hatcher - Wagoner

family. Quinn observed that after some modification of the 1-parameter family, analogous to reordering critical points in the proof of the h-cobordism theorem, the essential information of the family can be succinctly captured in what we call the *middle middle level picture*, see p. 353-354 [Qu]. While Quinn did this for the innermost eye, one readily proves the general case which we now state. Start with  $M_k$ , where  $k$  is the number of eye components, each  $S^2 \times S^2$  summand is standardly parametrized and  $(x_0, y_0) \in S^2 \times S^2$ . Call the  $x_0 \times S^2$  2-spheres the *standard red* spheres  $\mathcal{R}^{\text{std}} := \{R_1^{\text{std}}, \dots, R_k^{\text{std}}\}$  and the  $S^2 \times y_0$  2-spheres the *standard green* spheres  $\mathcal{G}^{\text{std}} := \{G_1^{\text{std}}, \dots, G_k^{\text{std}}\}$ . Now do a finite sequence of pairwise disjoint finger moves to the red spheres to obtain  $\mathcal{R} = \{R_1, \dots, R_k\}$  with new intersections with  $\mathcal{G}^{\text{std}}$ , one pair for each finger move. Suppose that there is a set of pairwise disjoint Whitney discs such that applying the corresponding Whitney moves to  $\mathcal{R}$  yields a new system of red spheres that  $\delta_{ij}$  pairwise geometrically intersect the components of  $\mathcal{G}^{\text{std}}$ . The middle middle level picture is the situation after the finger moves. Thus in  $M_k$  we have two sets of spheres  $\{R_1, \dots, R_k\}, \{G_1^{\text{std}}, \dots, G_k^{\text{std}}\}$  and two sets  $\mathcal{F} = \{f_p\}, \mathcal{W} = \{w_q\}$  of *Whitney discs* that cancel the excess  $R_i/G_j^{\text{std}}$  intersections. The  $\mathcal{F}$  Whitney discs are called *finger discs*. Doing Whitney moves using these discs undoes the finger moves. Starting with the middle middle level picture we create a 1-parameter family of maps  $q_t : M \times [0, 1] \times t \rightarrow [0, 1]$  and glvf's  $v_t, t \in [0, 1]$ , where  $q_{1/2}$  has  $k$  critical points of index-2 and  $k$  critical points of index-3,  $G_j^{\text{std}} \subset q_{1/2}^{-1}(1/2)$  is the ascending sphere of the  $j$ 'th index-2 critical point and  $R_i \subset q_{1/2}^{-1}(1/2)$  is the descending sphere of the  $i$ 'th index-3 critical point. Using  $\{w_q\}$  the handle structure on  $M \times I \times 1/2$  is modified, as is the corresponding  $(q_t, v_t)$ , to one whose ascending and descending spheres intersect geometrically  $\delta_{ij}$ . The index-2 and index-3 critical points are then cancelled at death critical points after which  $q_t$  is nonsingular. These modifications enable an extension of  $(q_{1/2}, v_{1/2})$  to  $(q_t, v_t), t \in [1/2, 1]$ . Similarly  $\{f_p\}$  enables an extension  $(q_t, v_t), t \in [0, 1/2]$ . This can be done so that the resulting  $(q_t, v_t)$  is a Hatcher - Wagoner family and the resulting pseudo isotopy is isotopic to  $f$ .

**Remarks 2.4.** i) See Chapter 1 [HW] for basic facts about 1-parameter families including descriptions of their low dimensional strata as well as terminology used in this section.

ii) Using the light bulb theorem [Ga1], [ST] the red spheres obtained by doing the Whitney moves can be isotoped back to  $\mathcal{R}^{\text{std}}$  and thus the pseudo-isotopy is determined, up to isotopy, by a loop in the embedding space of red spheres. Our original motivation for proving the light bulb theorem was constructing such a loop when  $M = S^1 \times S^3$ .

When  $\pi_1(M) = 1$ , Quinn showed that after finitely many stabilizations of the pseudo-isotopy  $f$  and modification of the glvf, the ascending and descending spheres from the critical points of the innermost eye component have no excess intersections. That component can be eliminated by Cerf's *unicity of death lemma* Chapter 3 [Ce3], [Ch] or P. 170 [HW]. By *stabilization of the pseudo-isotopy*  $f$  we mean first isotope  $f$  to be id on a  $B^4 \times I$ , then replace the  $B^4 \times I$  with a  $(B^4 \# S^2 \times S^2) \times I$  and extend  $f$  to be the identity on the  $(B^4 \# S^2 \times S^2) \times I$ . Finally, by induction on components, a 1-parameter family  $q_t$  can be constructed having no critical points and hence a sufficiently stabilized  $f$  is isotopic to id and therefore so is a stabilized  $\phi$ .

The following is the main result of this section. Its proof will be crucial for applications.

**Theorem 2.5.** *Let  $\phi \in \text{Diff}_0(M)$ , where  $M$  is a compact oriented 4-manifold. Then  $\text{id}$  is stably isotopic to  $\phi$  if and only if  $\text{id}$  is pseudo-isotopic to  $\phi$  by a pseudo-isotopy  $f$  with  $\Sigma(f) = 0$ .*

We start with the following well known result.

**Lemma 2.6.** *Let  $M$  be a compact 4-manifold and  $(q_t, v_t), (q'_t, v'_t)$  be two Hatcher - Wagoner 1-parameter families. If for some  $s \in [0, 1]$ , not a birth or death point,  $q_t = q'_t$  and  $v_t = v'_t$  for  $t \leq s$  and neither  $(q_t, v_t)$  nor  $(q'_t, v'_t)$  have excess  $3/2$  intersections for  $t \geq s$ , then the associated pseudo-isotopies  $f, f'$  are isotopic.*

*Proof.* We will assume that  $s = 1/2$ , is after all the births and before the deaths. The proof in the general case is similar. The 1-parameter family  $r_t := q_{1-t}$  for  $t \leq 1/2$  and  $r_t := q'_t$  for  $t \geq 1/2$  with corresponding glvf's is of the nested eye type without handle slides or excess  $3/2$  intersections, hence the singular locus can be eliminated by uniqueness of death. Thus induced pseudo-isotopy  $g$  is isotopic to  $\text{id}$ . Since  $f' = g \circ f$ , the result follows.  $\square$

*Proof of Theorem 2.5:* First assume that  $\text{id}$  is pseudo-isotopic to  $\phi$  by the pseudo-isotopy  $f$  with  $\Sigma(f) = 0$ , hence is realized by a Hatcher - Wagoner 1-parameter family  $(q_t, v_t)$ .

Using elements of the proof of Theorem 9 [Gay] we can arrange the following. For  $t \in [0, 1/8] \cup [7/8, 1]$ ,  $q_t$  is non singular. For  $t \in (1/8, 1/4)$  (resp.  $(3/4, 7/8)$ )  $k$  births (resp. deaths) occur. All the excess  $3/2$  intersections occur when  $t \in (1/4, 3/4)$ . For  $t \in [1/4, 3/4]$ ,  $q_t = q_{1/4}$ . Also, for such  $t$ ,  $v_t = v_{1/4}$  when restricted to  $q_t^{-1}([0, 1/4] \cup [3/4, 1])$ . The  $k$  critical points of index-2 (resp. index-3) lie in  $q_t^{-1}(1/8, 1/4)$  resp.  $q_t^{-1}(3/4, 7/8)$  all with distinct critical values. This requires a bit of preliminary work, e.g. as noted in [Gay] the descending spheres of the 2-handles in  $M \times 0 \times t$  are simple closed curves that may follow non trivial paths in  $\text{Emb}(S^1, M)$ , however it can be arranged that these paths are constant for  $t \in [1/4, 3/4]$  and all the movement shoved into  $t \in (3/4, 3/4 + \epsilon)$ . We can assume that  $v_{1/4}$  is the model gradient like vector field on  $M \times I \times 1/4$  arising from the births and has the following features. Both  $v_{1/4}^{-1}(1/4)$  and  $v_{1/4}^{-1}(3/4)$  are diffeomorphic to  $M_k$  and respectively denoted  $M_k \times 1/4 \times 1/4$  and  $M_k \times 3/4 \times 1/4$  with the first factors equated via  $v_{1/4}$ . Let  $G_i^{\text{std}} \subset M_k \times 1/4 \times 1/4$  (resp.  $R_j^{\text{std}} \subset M_k \times 3/4 \times 1/4$ ) denote the ascending (resp. descending) sphere of the  $i$ 'th 2-handle (resp.  $j$ 'th 3-handle). Under  $v_{1/4}$  the  $G_i^{\text{std}}$ 's flow to spheres in  $M_k \times 3/4 \times 1/4$  that  $\delta_{ij}$  intersect the  $R_j^{\text{std}}$ 's. Under  $v_{1/4}$  (resp.  $-v_{1/4}$ ) the  $G_i^{\text{std}}$ 's (resp.  $R_j^{\text{std}}$ 's) flow to discs  $\{E_1, \dots, E_k\} \subset M \times 1 \times 1/4$  (resp.  $\{D_1, \dots, D_k\} \subset M \times 0 \times 1/4$ ) spanning the ascending (resp. descending) spheres of the 3-handles (resp. 2-handles). When projected to  $M$  the  $D_i$ 's intersect  $\delta_{ij}$  the  $E_j$ 's and if  $N_i$  denotes a neighborhood of  $D_i \cup E_i$ , then  $v_{1/4}|(M \setminus \cup_{i=1}^k \text{int}(N_i)) \times I = v_0$ . Use  $v_{1/4}$  to define a product structure on  $q_{1/4}^{-1}([1/4, 3/4])$ . The induced map  $M_k \times 1/4 \times 1/4 \rightarrow M_k \times 3/4 \times 1/4$  when projected to the first factor is  $\text{id}$  and denoted  $\Phi_{1/4}$ . It is constructed from the  $\text{id}$  on  $M$  (which is induced from  $v_0$ ), replacing each  $N_i$  by an  $S^2 \times S^2 \setminus \text{int}(B^4)$  and then extending the  $\text{id}$ , i.e. the map  $\Phi_{1/4}$  on  $M_k$  is a stabilization of  $\text{id}$  on  $M$ .

We now show how to *remember* the  $\text{id}$  pseudo-isotopy as we pass from  $(q_0, v_0)$  to  $(q_1, v_1)$  and use it to construct a stable isotopy  $\Phi$  from  $\text{id}$  to a  $\phi'$ . We will show that  $\phi'$  is isotopic to  $\phi$  in the next paragraph. Since  $q_t = q_{1/4}$  for  $t \in [1/4, 3/4]$ ,  $\cup_{t \in [1/4, 3/4]} q_t^{-1}([1/4, 3/4])$  is diffeomorphic to  $M_k \times [1/4, 3/4] \times [1/4, 3/4] := \mathcal{C}$ , the *core* of the 1-parameter family. The core has two natural parameterizations, the first using the glvf  $v_{1/4}$  on each  $q_t^{-1}([1/4, 3/4])$

which we call the *standard parametrization* and a second using  $v_t$  on  $q_t^{-1}([1/4, 3/4])$  which we call the  $\Psi$  *parametrization*. Having defined  $G_i^{\text{std}} \times 1/4 \times 1/4$  and  $R_j^{\text{std}} \times 3/4 \times 1/4$  the standard parametrization allows us to identify the spheres  $G_i^{\text{std}} \times s \times t$  and  $R_j^{\text{std}} \times s \times t$ . Since  $v_{1/4}$  and  $v_t$  agree on  $q_t^{-1}([0, 1/4])$  we canonically identify  $\cup_{t \in [1/4, 3/4]} q_t^{-1}(1/4)$  with  $M_k \times 1/4 \times [1/4, 3/4]$ . We define the parametrization  $\Psi : \mathcal{C} \rightarrow \mathcal{C}$  by  $\Psi|_{M_k \times 1/4 \times [1/4, 3/4]} = \text{id}$ ,  $\Psi|_{M_k \times [1/4, 3/4] \times 1/4} = \text{id}$ ,  $\Psi$  fixes each  $M_k \times s \times t$  setwise and  $\Psi$  takes  $v_{1/4}$  flow lines to  $v_t$  flow lines. Since  $v_{1/4}$  and  $v_t$  agree on  $q_t^{-1}[3/4, 1]$ , each  $q_{s_0}^{-1}(3/4)$  is canonically identified with  $q_{s_1}^{-1}(3/4)$  for  $s_0, s_1 \in [1/4, 3/4]$  and these identifications agree with the ones given by the standard parametrization. In particular,  $R_j^{\text{std}} \times 3/4 \times 3/4$  is the descending sphere of the  $j$ 'th 3-handle.

Since  $\Psi^{-1}(R_j^{\text{std}} \times 3/4 \times 3/4)$  has geometric  $\delta_{ij}$  intersection with  $G_i^{\text{std}} \times 3/4 \times 3/4$ , we can isotope  $\Psi^{-1}|_{R_j^{\text{std}} \times 3/4 \times 3/4, j = 1, \dots, k}$  to  $\text{id}$  via an isotopy staying transverse to  $\cup G_i^{\text{std}} \times 3/4 \times 3/4$  by the light bulb theorem [Ga1], [ST]. It follows that we can homotope the  $v_t$ 's  $t \in (3/4 - \alpha/2, 3/4 + \alpha/2)$ , keeping  $v_t = v_{1/4}$  on  $q_t^{-1}([0, 1/4] \cup [3/4, 1])$ , such that with respect to the new  $v_t$ 's,  $\Psi^{-1}|_{R_j^{\text{std}} \times 3/4 \times 3/4} = \text{id}$ . Also, no new intersections between the ascending and descending spheres of the critical points are created. Continuing to call our new glvf family  $v_t$ , the resulting pseudo-isotopy  $f$  is unchanged. Note that when  $\pi_1(M)$  has 2-torsion [ST] applies since  $\Psi^{-1}(R_j^{\text{std}} \times 3/4 \times 3/4)$  is isotopic to  $R_j^{\text{std}} \times 3/4 \times 3/4$  in  $M_k \times 3/4 \times 3/4$  and so the Freedman-Quinn obstruction = 0. After a second application of the light bulb theorem we can additionally assume that  $\Psi|_{\cup G_i^{\text{std}} \times 3/4 \times 3/4} = \text{id}$ . Here we use the fact that homotopy implies isotopy keeping the dual sphere fixed pointwise provided that the intersection is preserved under the original map. Using uniqueness of regular neighborhoods we can further assume that

$$(*) \quad \Psi|_{\cup_{i=1}^k N(G_i^{\text{std}} \cup R_i^{\text{std}}) \times 3/4 \times 3/4} = \text{id}$$

where for  $s \in [1/4, 3/4]$ ,  $N(G_i^{\text{std}} \cup R_i^{\text{std}}) \times s \times 3/4 = q_{3/4}^{-1}(s) \cap N_i \times I \times 3/4$ . Define  $\Phi_s = \Psi|_{M_k \times s \times 3/4, s \in [1/4, 3/4]}$ , viewed as a map from  $M_k$  to  $M_k$ . It is a stable isotopy from  $\text{id}$  to some  $\phi' \in \text{Diff}(M)$ . Call a vector field  $v_t$  as above satisfying  $(*)$  *stable-inducing*.

We now show that  $\phi'$  is isotopic to  $\phi$ . Define  $\hat{\Psi} : M \times I \times 3/4 \rightarrow M \times I \times 3/4$  by  $\hat{\Psi}|_{q_{3/4}^{-1}([0, 1/4])} = \text{id}$ ,  $\hat{\Psi}|_{q_{3/4}^{-1}([1/4, 3/4])} = \Psi$ ,  $\hat{\Psi}|_{(q_{3/4}^{-1}([3/4, 1]) \cap (N_i \times I))} = \text{id}$  all  $i$  and  $\hat{\Psi}|_{q_{3/4}^{-1}([3/4, 1]) \cap (M \times I \setminus (\text{int}(\cup N_i) \times I))}$  the extension of  $\Psi$  which takes  $v_{1/4}$  flow lines to  $v_{3/4}$  flow lines. Recall that  $v_{1/4} = v_{3/4}$  in that region. Next define a 1-parameter family  $(q'_t, v'_t), t \in [0, 1]$  by  $(q'_t, v'_t) = (q_t, v_t)$  for  $t \in [0, 3/4]$  and for  $t \in [3/4, 1]$  first define  $\hat{\Psi}_t : M \times I \times (1-t) \rightarrow M \times I \times t$  to be the map which agrees with  $\hat{\Psi}$  on the  $M \times I$  factor. Next define  $q'_t = q_{1-t} \circ \hat{\Psi}_t^{-1}$  and  $v'_t = (\hat{\Psi}_t)_*(v_{1-t})$ . By construction the pseudo-isotopy  $f'$  arising from  $(q'_t, v'_t)$  is from  $\text{id}$  to  $\phi'$ . By Lemma 2.6,  $\phi'$  is isotopic to  $\phi$ .

Conversely, suppose that we are given a stable isotopy  $\Phi_s, s \in [1/4, 3/4]$  from  $\text{id}$  to  $\phi$ . We can assume that  $\Phi_s = \Phi_{1/4}$  (resp.  $\Phi_{3/4}$ ) for  $s$   $\epsilon$ -close to  $1/4$  (resp.  $3/4$ ). Construct a 1-parameter family  $(p_t, \omega_t), t \in [0, 1]$  as follows. First, for  $t \in [0, 3/4]$  let  $q_t$  be as above and define  $p_t = q_t$ . Next define a vector field  $\omega$  on  $M \times I \times [0, 3/4]$  which restricts to a glvf  $\omega_t$  on each  $M \times I \times t$  as follows. For  $t \in [0, 1/4]$  let  $\omega_t = v_t$ . Also for  $t \in [1/4, 3/4]$  define  $\omega_t = v_{1/4}$  when restricted to  $p_t^{-1}([0, 1/4] \cup [3/4, 1])$ . For  $t \in [3/4 - \epsilon, 3/4]$  define  $\omega_t$  along  $M_k \times [1/4, 3/4] \times t$  by  $\Phi_*(v_{1/4})$ , where given  $t$ , define  $\Phi(x, s, 1/4) = (\Phi_s(x), s, t)$ . Use a partition of unity argument to extend our partially defined  $\omega$  to a vector field on



$M \times I \times [0, 3/4]$  which restricts to a glvf on each  $M \times I \times t$ . As in the proof of the forward direction, extend the 1-parameter family to  $t \in [0, 1]$  to obtain a Hatcher - Wagoner family yielding a pseudo-isotopy from id to  $\phi$ .  $\square$

**Definition 2.7.** The operation of obtaining a stable isotopy (resp. 1-parameter family) from a 1-parameter family (resp. stable isotopy) as above is called *extracting* (resp. *transplanting*).

**Remark 2.8.** Our stable isotopy is found by peering inside  $(M \times I) \times I$ . In contrast, assuming  $\pi_1(M) = 1$ , Quinn stabilizes the pseudo-isotopy  $f$  itself and then inductively modifies the 1-parameter family to one without critical points to turn the pseudo-isotopy into an isotopy.

**Definition 2.9.** Let  $\text{PI}(M)$  denote the group of isotopy classes of pseudo-isotopies of  $M$  starting at id. Let  $\text{PI}^\Sigma(M)$  denote the subgroup of classes  $f$  with  $\Sigma(f) = 0$ . Let  $\text{Diff}_0^\Sigma(M)$  denote the subgroup of  $\text{Diff}_0(M)$  generated by elements pseudo-isotopic to id by a pseudo-isotopy  $f$  with  $\Sigma(f) = 0$ .

**Remark 2.10.** By [HW] p.12,  $\text{Wh}_2(G) = 0$  if  $G$  is either free abelian or free, hence if  $\pi_1(M)$  is either free or free abelian, then by [HW]  $\Sigma(f) = 0$  for all pseudo-isotopies of  $M$ . Therefore, for such  $M$   $\text{Diff}_0(M) = \text{Diff}_0^\Sigma(M)$ .

**Corollary 2.11.** *If  $\phi \in \text{Diff}_0(S^1 \times S^3)$ , then  $\phi$  is stably isotopic to id.*

*Proof.* In 1968 Lashoff - Shaneson [LS] and Sato [Sa] proved that if  $\phi \in \text{Diff}_0(S^1 \times S^q)$ ,  $q = 3, 4$  then  $\phi$  is pseudo-isotopic to id. Now apply Remark 2.10 and Theorem 2.5.  $\square$

**Remark 2.12.** The punchline of Sato's proof uses Theorem III from the Appendix of [Ke] to prove that a certain homotopy  $q+1$ -sphere  $\Sigma^{q+1}$  is smoothly standard, but Theorem III is not applicable here for dimensional reasons. However, he shows that  $\Sigma^{q+1}$  is the union of two  $B^{q+1}$ 's glued along their boundary, hence is standard by Cerf [Ce2]) when  $q = 3$  or Kervaire - Milnor [KM] and Smale [Sm2] when  $q = 4$ .

**Corollary 2.13.** *If  $\phi \in \text{Diff}_0(M)$  and  $\pi_1(M)$  is either free or free abelian, then  $\phi$  is pseudo-isotopic to id if and only if  $\phi$  is stably isotopic to id.*  $\square$

**Question 2.14.** *If  $\Sigma(f) = 0$  is  $f$  stably isotopic to id?*

The following result was independently proven by David Gay [Gay]. He stated it for  $M = S^4$ , however his proof extends verbatim to the generality below. See also [KK] when  $M$  is simply connected.

**Theorem 2.15.** *Let  $M$  be a compact oriented 4-manifold. Let  $\mathcal{R}_k$  denote a disjoint union of  $k$  ordered 2-spheres.*

- i) There is a homomorphism  $\gamma_k : \pi_1(\text{Emb}(\mathcal{R}_k, M_k; \mathcal{R}^{\text{std}})) \rightarrow \pi_0(\text{Diff}_0^\Sigma(M))$ .*
- ii)  $\pi_0(\text{Diff}_0^\Sigma(M)) = \cup_{k=1}^\infty \text{Im}(\gamma_k)$ .*
- iii) If  $[\alpha] \in \pi_1(\text{Emb}(\mathcal{R}_k, M_k; \mathcal{R}^{\text{std}}))$  and  $[\beta] = [h * \alpha * k]$  where for all  $t$ ,  $h_t(\mathcal{R}_k) \cap \mathcal{G}^{\text{std}} = k_t(\mathcal{R}_k) \cap \mathcal{G}^{\text{std}} = \delta_{ij}$ , then  $\gamma_k[\alpha] = \gamma_k[\beta]$ .*

*Proof.* We abuse notation by identifying  $\mathcal{R}_k$  with  $\mathcal{R}^{\text{std}}$ . Let  $\alpha_t : \mathcal{R}^{\text{std}} \rightarrow M_k$  with  $\alpha_0 = \alpha_1 = \text{id}$ . Use parametrized isotopy extension to obtain the extension  $\alpha_t : \mathcal{R}^{\text{std}} \cup \mathcal{G}^{\text{std}} \rightarrow M_k$ . After a further isotopy fixing  $\mathcal{R}^{\text{std}}$  pointwise, we can assume that  $\alpha_1$  restricts to id near  $\mathcal{R}^{\text{std}} \cap \mathcal{G}^{\text{std}}$ . By [Gal], [ST] and uniqueness of regular neighborhoods we can further assume

that  $\alpha_1|N(\mathcal{R}^{\text{std}} \cup \mathcal{G}^{\text{std}}) = \text{id}$  without modifying  $\alpha_1|\mathcal{R}^{\text{std}}$ . Thus  $\alpha_t$  extends to a stable isotopy of  $\text{id}$  and hence induces a  $\phi \in \text{Diff}_0(M)$ . The isotopy class of  $\phi$  is independent of choices since composing one with the inverse of the other produces a stable isotopy isotopic to one which fixes  $\mathcal{R}^{\text{std}}$  and hence induces a diffeomorphism isotopic to  $\text{id}$  by Lemma 2.6. It similarly follows that the isotopy class is independent of the representative of  $[\alpha]$ . Since the concatenation of loops gives rise to the composition of stable isotopies the proof of i) is complete. Given  $\phi \in \pi_0(\text{Diff}_0^\Sigma(M))$ , construct a pseudo-isotopy from  $\text{id}$  to  $\phi$  and then extract a stable isotopy  $\Phi$  from an associated 1-parameter family to produce an  $\alpha$  with  $\gamma_k([\alpha]) = \phi$ . Here  $\alpha_s = \Phi_s|\mathcal{R}^{\text{std}}$ . Finally for iii), create 1-parameter families by transplanting stable isotopies arising from  $\alpha$  and  $\beta$ , then use Lemma 2.6 to conclude that the associated pseudo-isotopies are isotopic and hence so are the induced elements of  $\text{Diff}_0(M)$ .  $\square$

**Remarks 2.16.** i) There is the analogous result with  $\mathcal{G}^{\text{std}}$  in place of  $\mathcal{R}^{\text{std}}$ .

ii) The original motivation to prove the light bulb theorem was to prove i), ii) for  $M = S^1 \times S^3$ .

**Definition 2.17.** Let  $\pi_{M_k} : M_k \times [1/4, 3/4] \times [1/4, 3/4] \rightarrow M_k$  be the projection. Let  $v_t$  be a stable-inducing vector field inducing the parametrization  $\Psi$  on the core. Define  $\Psi_T : M_k \times [1/4, 3/4] \times [1/4, 3/4] \rightarrow M_k \times [1/4, 3/4] \times [1/4, 3/4]$ , the  $\Psi_T$ -parametrization by  $\Psi_T|M_k \times 3/4 \times [1/4, 3/4] = \text{id}$ ,  $\Psi_T$  fixes each  $M_k \times s \times t$  setwise and  $\Psi_T$  takes  $v_{1/4}$  flow lines to  $v_t$  flow lines. Define  $\alpha_{s,t} = \pi_{M_k} \circ \Psi_T|\mathcal{R}^{\text{std}} \times s \times t$ .

**Lemma 2.18.** *If  $\Psi$  and  $\Psi_T$  are the parametrizations of the core of a  $\Phi$ -transplanted stable-isotopy of the compact 4-manifold  $M$ , then  $\alpha_s = \alpha_{s,3/4}$ ,  $s \in [1/4, 3/4]$ . As a based loop  $\alpha_s$  is homotopic to  $\alpha_{1/4,1-s}$ . The glvf  $\omega_t$  can be chosen so that these loops are  $C^1$ -close.*

*Proof.* Since  $v_t$  is stable-inducing, for  $s \in [1/4, 3/4]$ ,  $\Psi|N(\mathcal{R}^{\text{std}} \cup \mathcal{G}^{\text{std}}) \times s \times 3/4 = \Psi_T|N(\mathcal{R}^{\text{std}} \cup \mathcal{G}^{\text{std}}) \times s \times 3/4$  and hence  $\alpha_s = \Phi_s|\mathcal{R}^{\text{std}} = \pi_{M_k} \circ \Psi|\mathcal{R}^{\text{std}} \times s \times 3/4 = \pi_{M_k} \circ \Psi_T|\mathcal{R}^{\text{std}} \times s \times 3/4 = \alpha_{s,3/4}$ . Now  $\alpha_{s,3/4}$  is homotopic to  $\alpha_{1/4,1-s} * \alpha_{u,1/4} * \alpha_{3/4,v}$ , where  $u, v \in [1/4, 3/4]$ . Note that these last two paths are the  $\text{id}$  on  $\mathcal{R}^{\text{std}}$ . Now define the 1-parameter family of glvfs  $w_t$  on  $M_k \times [1/4, 3/4] \times [1/4, 3/4]$  so that  $w_t$  agrees with  $v_{3/4}$  on  $M_k \times s \times t$  when  $s \geq 1 - t$ , agrees with  $v_{1/4}$  when  $s \leq 1 - t$  and then use a partition of unity to smooth out along  $M_k \times (1 - t) \times t$ . Since  $\alpha_{1/4,1-t}$  is approximately  $\alpha_{t,1-t}$  which is approximately  $\alpha_{t,3/4} = \alpha_t$ , the result follows.  $\square$

**Remark 2.19.** The path  $\alpha_{1/4,t}$ ,  $t \in [1/4, 3/4]$  is the result of flowing the descending sphere  $\mathcal{R}^{\text{std}} \times 3/4 \times t$  into  $M_k \times 1/4 \times t$  using the glvf  $\omega_t$  and then projecting to the first factor. This path is homotopic to the reverse of  $\alpha_s$  arising from the stable isotopy which we can assume is  $C^1$ -close to it.

### 3. FINGER - WHITNEY SYSTEMS

Let  $\Phi : M_k \times I \rightarrow M_k$  be a stable isotopy from  $\text{id}$  to  $\phi \in \text{Diff}_0(M)$  and let  $\mathcal{R}^{\text{std}}, \mathcal{G}^{\text{std}}$  be defined as in §2. We can assume that  $\Phi_s(\mathcal{R}^{\text{std}})$  is transverse to  $\mathcal{G}^{\text{std}}$  except for finitely many  $s \in I$  corresponding to finger and Whitney moves and all the finger moves occur before the Whitney moves. For more details see [FQ] pp. 19-20 and [Qu] p. 353.

**Definition 3.1.** A *finger-Whitney* or *F|W system*  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  on  $M_k$  consists of two transverse sets of algebraically dual, pairwise disjoint embedded 2-spheres  $\mathcal{G} := \{G_j\}$ ,  $\mathcal{R} := \{R_i\}$ ,

(i.e.  $\langle R_i, G_j \rangle = \delta_{ij}$ ) with trivial normal bundles in  $M_k$  together with two complete collections of Whitney discs  $\mathcal{F} := \{f_p\}, \mathcal{W} := \{w_q\}$ . This means that performing Whitney moves to  $\mathcal{R}$  using either set of discs produces a set of spheres geometrically dual to  $\mathcal{G}$ . We require that  $M$  is diffeomorphic to the manifold obtained by surgering  $M_k$  along the  $\mathcal{G}$  family, i.e. replacing neighborhoods of the  $G_i$ 's by  $S^1 \times B^3$ 's. Call  $\mathcal{F}$  the set of *finger discs* and  $\mathcal{W}$  the *Whitney discs*. Call a  $F|W$  system arising from a stable isotopy of  $M$  as in the introduction an *induced  $F|W$  system*

**Remark 3.2.** Finger-Whitney systems have their origin in §4 [Qu], where  $\mathcal{G}$  and  $\mathcal{R}$  are as in the middle middle level picture. With the conventions of this paper the roles of finger and Whitney discs are reversed. See Remark 2.19 and the proof of the following.

**Lemma 3.3.** *Let  $M$  be a compact orientable 4-manifold. A  $F|W$  system on  $M_k$  determines a conjugacy class  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}) \in \pi_0(\text{Diff}_0(M))$ . Conversely, given  $\phi \in \pi_0(\text{Diff}_0^\Sigma(M))$ , there exists an  $F|W$  system on some  $M_k$  inducing  $\phi$ .*

*Proof.* Let  $\mathcal{R}'$  (resp.  $\mathcal{R}''$ ) be obtained from  $\mathcal{R}$  by doing the Whitney moves  $\mathcal{F}$  (resp.  $\mathcal{W}$ ). Since  $M$  is diffeomorphic to  $M_k$  surgered along  $\mathcal{G}$ , there is a  $\zeta_k \in \text{Diff}(M_k)$  such that  $\zeta_k(\mathcal{R}^{\text{std}}) = \mathcal{R}'$  and  $\zeta_k(\mathcal{G}^{\text{std}}) = \mathcal{G}$ . To see this note that after surgery the components of  $\mathcal{G}^{\text{std}}$  and  $\mathcal{G}$  become circles respectively spanned by 0-framed discs arising from  $\mathcal{R}^{\text{std}}$  and  $\mathcal{R}'$  and that a  $\zeta \in \text{Diff}_0(M)$  takes one set to the other. It therefore suffices to consider the case that  $(\mathcal{G}, \mathcal{R}') = (\mathcal{G}^{\text{std}}, \mathcal{R}^{\text{std}})$ .

The  $F|W$  system gives rise to a loop  $\alpha_t \in \text{Emb}(\mathcal{R}_k, M_k; \mathcal{R}^{\text{std}})$  obtained by first isotoping to  $\mathcal{R}$  by the finger moves, then to  $\mathcal{R}''$  by the Whitney moves and then back to  $\mathcal{R}^{\text{std}}$  by an isotopy fixing  $\mathcal{G}$  setwise. The last isotopy follows from [Ga1] or [ST]. Again, since  $\mathcal{R}''$  is isotopic to  $\mathcal{R}^{\text{std}} \subset M_k$ , it follows that  $\text{FQ}(\mathcal{R}^{\text{std}}, \mathcal{R}'') = 0$ . By Theorem 2.15  $\alpha_t$  induces an element of  $\pi_0(\text{Diff}_0(M))$ . Note that different identifications of  $(\mathcal{G}, \mathcal{R}')$  with  $(\mathcal{G}^{\text{std}}, \mathcal{R}^{\text{std}})$ , using the method above, will change the resulting class in  $\pi_0(\text{Diff}_0(M))$  by conjugation in  $\pi_0(\text{Diff}_0(M))$ .

Conversely, if  $\Phi$  is a stable isotopy of id to  $\phi$ , then after isotopy we can assume that  $\Phi_s(\mathcal{R}^{\text{std}})$  is a generic isotopy such that all the finger (resp. Whitney) moves with  $\mathcal{G}^{\text{std}}$  occur before (resp. after)  $s = 1/2$ . It follows that  $(\mathcal{G}^{\text{std}}, \Phi_{1/2}(\mathcal{R}^{\text{std}}), \mathcal{F}, \mathcal{W})$  is a  $F|W$  system on  $M_k$ , inducing  $\phi$  where  $\mathcal{W}$  (resp.  $\mathcal{F}$ ) is the set of Whitney (resp. finger) discs.  $\square$

**Corollary 3.4.** *A  $F|W$  system on  $M$  induces a stable isotopy whose associated element of  $\pi_0(\text{Diff}_0(M))$  is unique up to conjugacy.*  $\square$

**Corollary 3.5.** *If  $\phi \in \pi_0(\text{Diff}_0(S^1 \times S^3))$ , then there exists a  $F|W$  system inducing  $\phi$ . A given  $F|W$  system on  $S^1 \times S^3$  induces a well defined element of  $\pi_0(\text{Diff}_0(S^1 \times S^3))$ .*

*Proof.* Since  $\text{Diff}_0(S^1 \times S^3)$  is abelian, the second statement follows from Lemma 3.3. The first follows from the fact that  $\pi_0(\text{Diff}_0(S^1 \times S^3)) = \pi_0(\text{Diff}_0^\Sigma(S^1 \times S^3))$ .  $\square$

**Remark 3.6.** Our original argument for the existence of a  $F|W$  system for  $\phi \in \text{Diff}_0(S^1 \times S^3)$  is as follows. A given  $\phi$  is pseudo-isotopic to id by [LS], [Sa]. It has a Hatcher - Wagoner 1-parameter family by [HW]. Now apply Quinn's method as outlined in §2.

We now specialize to  $\phi \in \text{Diff}_0(S^1 \times S^3)$ , although much of what follows applies more generally. Denote  $S^1 \times S^3$  by  $V$ ,  $S^1 \times S^3 \#_k S^2 \times S^2$  by  $V_k$  and  $V_\infty := \mathbb{R} \times S^3 \#_\infty S^2 \times S^2$ , where the sums are locally finite and go out both ends. In practice  $V_\infty = \tilde{V}_k$  for some  $k$ .

**Notation 3.7.** Denote the  $S^2 \times S^2$  factors of  $V_k$  by  $S^2 \times S_i^2$ . By  $S^2 \times S_i^2$  we mean a  $S^2 \times S^2 \setminus \text{int } B^4$ , where  $B^4$  is disjoint from  $R_i^{\text{std}} \cup G_i^{\text{std}}$ . Denote by  $\Sigma_i$  the 3-sphere that separates off  $S^2 \times S_i^2$  from the rest. When  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is an  $F|W$  system on  $V_k$ , then we will assume that  $\mathcal{G} = \mathcal{G}^{\text{std}}$  and  $\mathcal{R}$  is isotopic to  $\mathcal{R}^{\text{std}}$ . Let  $Z = V_k \setminus \bigcup_{i=1}^k \text{int}(S^2 \times S_i^2)$ . Let  $\pi : Z \rightarrow S^1$  be the restriction of the projection map from  $S^1 \times S^3 \rightarrow S^1$ . Viewing  $S^1$  as  $[0, k]$  with 0 identified with  $k$ , we can assume that  $\pi(\Sigma_i) = N_\epsilon(i)$ . Indices are often chosen mod  $k$ . Here  $\tilde{\pi}, \tilde{Z}, \widetilde{S^2 \times S_j^2}, \tilde{R}_i, \tilde{G}_j$  denote fixed lifts to  $\tilde{V}_k$ , where the indices  $\in \mathbb{Z}$ . Let  $\rho : \mathbb{R} \times S^3 \rightarrow \mathbb{R} \times S^3$  be the covering translation that shifts everything  $k$  units, e.g.  $\rho(\tilde{R}_j) = \tilde{R}_{j+k}$ .

**Definition 3.8.** Let  $w \in \mathcal{F} \cup \mathcal{W}$ . We say the *winding*  $\omega(w) = r \in \mathbb{Q}$  if  $r = (q - p)/k$  where  $w$  lifts to a Whitney disc between  $\tilde{R}_p$  and  $\tilde{G}_q$ . Say that  $w$  and  $w'$  are *winding equivalent* if both  $w, w'$  connect  $R_i$  to  $G_j$  and  $\omega(w) = \omega(w')$ .

**Remark 3.9.** Winding depends on how  $\pi$  was chosen, however it is well defined when  $w$  is between  $R_i$  and  $G_i$ . Also, when  $w, w'$  are Whitney discs between  $R_i$  and  $G_j$ , then the truth of  $\omega(w) \neq \omega(w')$  is independent of  $\pi$ . It follows that the winding equivalence relation is well defined.

**Definition 3.10.** Let  $\mathcal{F} = \{f_1, \dots, f_p\}$  and  $\{[f_{i_1, j_1, \omega_1}], \dots, [f_{i_m, j_m, \omega_m}]\}$  the winding equivalence classes, where  $f_{i, j, \omega}$  denotes the class where a finger goes from  $R_i$  to  $G_j$  with winding  $\omega$ . We say that  $\mathcal{R}$  is in *arm hand finger (AHF) form* with respect to  $\mathcal{F}$  if it is constructed as follows. For the class  $[f_{i, j, \omega}]$  with  $|[f_{i, j, \omega}]|$  elements, remove a disc from  $R_i^{\text{std}}$  and replace it by a disc  $a_{i, j, \omega}$  called an *arm* which consists of a thin annulus  $\subset S^2 \times S_i^2$  that goes essentially straight from  $R_i$  to  $\Sigma_i$ , then a thin annulus  $\subset Z$  that goes essentially straight to  $\Sigma_j$  and winds  $\omega$  about  $Z$ , then a disc in  $S^2 \times S_j^2$  called a *hand* with  $|[f_{i, j, \omega}]|$  fingers. Finally,  $\mathcal{F}$  are the discs associated to the fingers. When  $i = j$  and  $\omega = 0$ , then  $a_{i, i, 0} \subset S^2 \times S_i^2$ . See Figure 1.

In similar manner define AHF form of  $\mathcal{R}$  with respect to  $\mathcal{W}$ . And in analogous manner define AHF form of  $\mathcal{G}^{\text{std}}$  with respect to either  $\mathcal{F}$  or  $\mathcal{W}$ . This form is the result of an ambient isotopy taking  $\mathcal{R}$  to  $\mathcal{R}^{\text{std}}$  and  $\mathcal{G}^{\text{std}}$  to  $\mathcal{G}$  where the latter is obtained from  $\mathcal{G}^{\text{std}}$  by attaching arms hands and fingers which poke into  $\mathcal{R}^{\text{std}}$ , where the discs corresponding to the fingers are the isotoped  $\mathcal{F}$  or  $\mathcal{W}$  discs as applicable.

**Lemma 3.11.**  $\mathcal{R}$  can be isotoped into AHF form with respect to  $\mathcal{F}$  as well as  $\mathcal{W}$  via an ambient isotopy that fixes  $\mathcal{G}^{\text{std}}$  setwise. Also  $\mathcal{G}^{\text{std}}$  can be isotoped into AHF form with respect to  $\mathcal{F}$  as well as  $\mathcal{W}$  via an ambient isotopy.

*Proof.* We show how to isotope  $\mathcal{R}$  into AHF form with respect to  $\mathcal{F}$ . There exists a set of pairwise disjoint *finger arcs* from  $\mathcal{R}^{\text{std}}$  to  $\mathcal{G}^{\text{std}}$  such that if  $\mathcal{R}_1$  is the result of doing finger moves to  $\mathcal{R}^{\text{std}}$  along these arcs and  $\mathcal{F}_1$  is the resulting set of finger discs, then  $(\mathcal{R}_1, \mathcal{F}_1)$  is isotopic to  $(\mathcal{R}, \mathcal{F})$  via an isotopy fixing  $\mathcal{G}^{\text{std}}$  setwise. To see this let  $\mathcal{R}'$  be the result of applying the  $\mathcal{F}$  Whitney moves to  $\mathcal{R}$  and noting  $(\mathcal{R}, \mathcal{F})$  arises from  $\mathcal{R}'$  by finger moves along finger arcs to  $\mathcal{G}^{\text{std}}$ . Now apply [Ga1] to isotope  $\mathcal{R}'$  to  $\mathcal{R}^{\text{std}}$  fixing  $\mathcal{G}^{\text{std}}$  setwise and consider the finger arcs resulting from applying isotopy extension to the finger arcs from  $\mathcal{R}'$ .

Since  $\pi_1(V_k \setminus \mathcal{G}^{\text{std}} \cup \mathcal{R}^{\text{std}}) = \mathbb{Z}$ , the result follows because we can isotope the finger arcs so that each goes from an  $R_i^{\text{std}}$  to  $\Sigma_i$  to  $\Sigma_j$  to  $G_j^{\text{std}}$  by essentially straight arcs with the correct winding and any two finger arcs corresponding to equivalent fingers are parallel.  $\square$

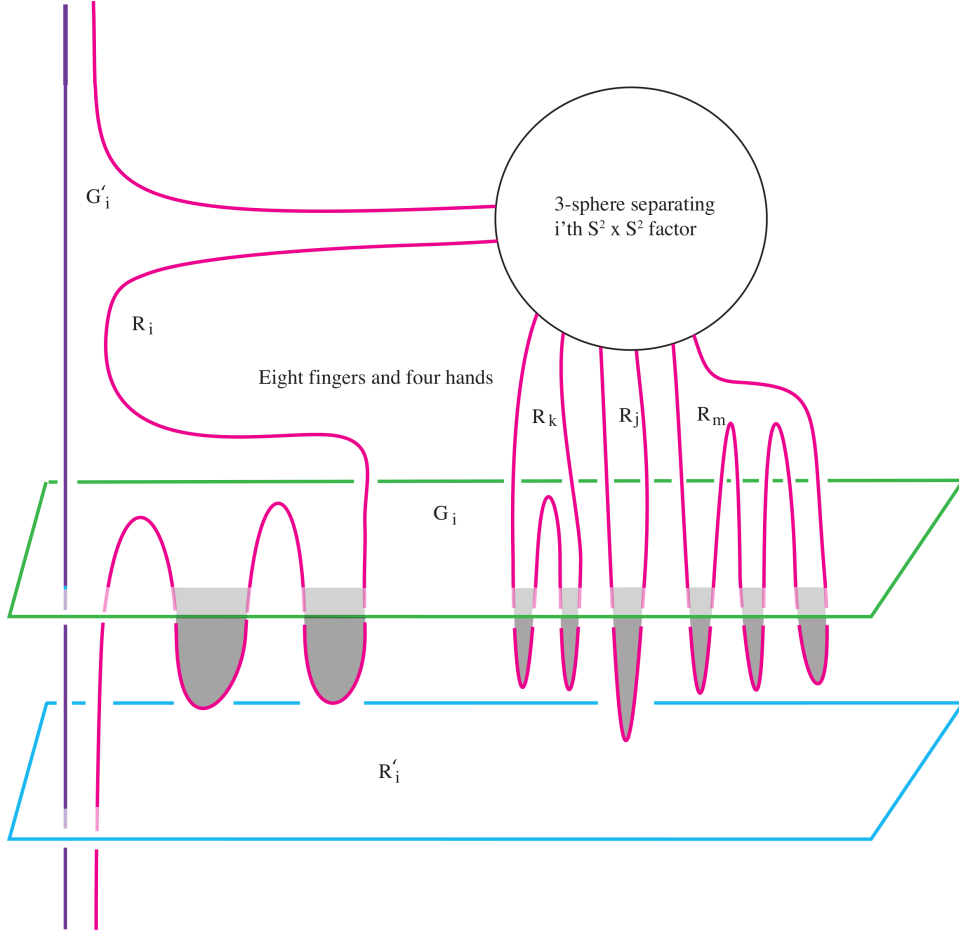


FIGURE 1. Arm Hand Finger Form

**Definition 3.12.** If  $\mathcal{R}$  is in arm hand finger form, then its *auxiliary discs* are the Whitney discs near the hands as in Figure 1. Note that a hand with  $f$  fingers has  $f - 1$  auxiliary discs.

**Definition 3.13.** ( $F|W$  Moves) We define the following operations on  $F|W$  systems.

- i) *reverse*:  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}) \rightarrow (\mathcal{G}, \mathcal{R}, \mathcal{W}, \mathcal{F})$ .
- ii) *upside down*:  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}) \rightarrow (\mathcal{R}, \mathcal{G}, \mathcal{W}, \mathcal{F})$
- iii) *concatenation*:  $(\mathcal{G}_1, \mathcal{R}_1, \mathcal{F}_1, \mathcal{W}_1) \cup (\mathcal{G}_2, \mathcal{R}_2, \mathcal{F}_2, \mathcal{W}_2)$ . This an  $F|W$  system on  $V_{k_1+k_2}$  where the first  $F|W$  system is on  $V_{k_1}$  and the second on  $V_{k_2}$ . Here view  $S^1 \times S^3 = S^1 \times B^3 \cup_{\partial} S^1 \times B^3$  with each system of the concatenation supported in its own  $S^1 \times B^3$ .

**Lemma 3.14.** ( $F|W$  Operations)

- i)  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{W}, \mathcal{F}) = (\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}))^{-1} := \phi^{-1}(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$
- ii)  $\phi(\mathcal{R}, \mathcal{G}, \mathcal{W}, \mathcal{F}) = \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$
- iii)  $\phi(\mathcal{G}_1, \mathcal{R}_1, \mathcal{F}_1, \mathcal{W}_1) \cup (\mathcal{G}_2, \mathcal{R}_2, \mathcal{F}_2, \mathcal{W}_2) = \phi(\mathcal{G}_1, \mathcal{R}_1, \mathcal{F}_1, \mathcal{W}_1) \circ \phi(\mathcal{G}_2, \mathcal{R}_2, \mathcal{F}_2, \mathcal{W}_2)$ .

*Proof.* i) If  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  arises from the 1-parameter family  $(q_t, v_t)$ , then  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{W}, \mathcal{F})$  arises from  $(q_{1-t}, v_{1-t})$ . This produces a pseudo-isotopy from id to  $\phi^{-1}$ .

ii) The loop  $\beta_s := \Phi_s|_{\mathcal{G}^{\text{std}}} = \pi_{V_k} \circ \Psi|_{\mathcal{G}^{\text{std}}} \times s \times 3/4$  induces  $\phi$ . Let  $\beta_{s,t} = \pi_{V_k} \circ \Psi|_{\mathcal{G}^{\text{std}}} \times s \times t$ . Then  $\beta_s = \beta_{s,3/4}$  is homotopic to  $\beta_{1/4,1-u} * \beta_{v,1/4} * \beta_{3/4,t}$ ,  $u, v, t \in [1/4, 3/4]$ , which is homotopic to  $\beta_{3/4,t}$  since the first two loops are constant. Under  $\omega_t$  when projected to  $V_k$ ,  $(\mathcal{G}^{\text{std}}, \alpha_{1/4,t}(\mathcal{R}^{\text{std}}))$  flows to  $(\beta_{3/4,t}(\mathcal{G}^{\text{std}}), \mathcal{R}^{\text{std}})$ . Since  $\alpha_{1/4,s}$  approximates  $\alpha_{1-s}$ , as  $s$  increases  $\alpha_{1/4,t}(\mathcal{R}^{\text{std}})$  undergoes inverse Whitney and then inverse finger moves corresponding to the  $F|W$  system associated to  $\alpha_s$ . This process flows under  $\omega_t$  to  $\mathcal{G}^{\text{std}}$  undergoing inverse Whitney moves and then inverse finger moves corresponding to the  $F|W$  system obtained by flowing the original system seen in  $V_k \times 1/4 \times 1/2$  to one in  $V_k \times 3/4 \times 1/2$ .

iii) The pseudo-isotopy arising from  $\phi(\mathcal{G}_1, \mathcal{R}_1, \mathcal{F}_1, \mathcal{W}_1) \circ \phi(\mathcal{G}_2, \mathcal{R}_2, \mathcal{F}_2, \mathcal{W}_2)$  is isotopic to the concatenation of the pseudo-isotopies from  $(\mathcal{G}_1, \mathcal{R}_1, \mathcal{F}_1, \mathcal{W}_1)$  and  $(\mathcal{G}_2, \mathcal{R}_2, \mathcal{F}_2, \mathcal{W}_2)$ . By *concatenation* we mean, supported on disjoint vertical  $S^1 \times B^3 \times I$ 's  $\subset S^1 \times S^3 \times I$ .  $\square$

**Lemma 3.15.** (*factorization lemma*) *If  $(\mathcal{G}, \mathcal{R}, \mathcal{F}_1, \mathcal{F}_2), (\mathcal{G}, \mathcal{R}, \mathcal{F}_2, \mathcal{F}_3)$  are  $F|W$  systems on  $V_k$ , then  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}_2, \mathcal{F}_3) \circ \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}_1, \mathcal{F}_2) = \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}_1, \mathcal{F}_3)$ .*

*Proof.* The map  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}_1, \mathcal{F}_2)$  is induced by the following loop  $\alpha_{12}$  in  $\text{Emb}(\mathcal{R}_1, V_k)$  where  $\mathcal{R}_1$  is the result of applying the  $\mathcal{F}_1$  Whitney moves to  $\mathcal{R}$ . First undo the  $\mathcal{F}_1$  moves to get back to  $\mathcal{R}$ , then do the  $\mathcal{F}_2$  Whitney moves to get  $\mathcal{R}_2$ , then isotope back to  $\mathcal{R}_1$ , say using the path  $\beta$  transverse to  $\mathcal{G}$ . The map  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}_2, \mathcal{F}_3)$  is induced by the loop  $\alpha_{23}$  defined by starting at  $\mathcal{R}_1$ , next applying the reverse of  $\beta$  to get back to  $\mathcal{R}_2$ , then applying the reverse  $\mathcal{F}_2$  moves to get back to  $\mathcal{R}$ , then applying the  $\mathcal{F}_3$  Whitney moves, then isotoping to  $\mathcal{R}_1$  via an isotopy transverse to  $\mathcal{G}$ . But  $\alpha_{12} * \alpha_{23}$  is homotopic to the loop  $\alpha_{13}$  that induces  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}_1, \mathcal{F}_3)$ .  $\square$

The next result follows from Proposition 1.12 and the methods of this section.

**Lemma 3.16.** *Let  $r$  denote a reflection of  $S^1 \times S^3$  across the  $S^3$  factor. If  $\mathcal{R}, \mathcal{G}$  and  $\mathcal{F}$  are invariant under  $r$ , then  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}) = \bar{\phi}(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  where  $\bar{\mathcal{W}} = r(\mathcal{W})$ .  $\square$*

#### 4. INTERPOLATION

Recall from §1 that  $\phi, \phi' \in \pi_0(\text{Diff}_0(S^1 \times S^3))$  are S-equivalent if they interpolate when lifted to  $\mathbb{R} \times S^3$ . In a similar manner we define S-equivalence for pseudo-isotopies, stable isotopies, 1-parameter families and  $F|W$  systems and show that S-equivalence for any two such structures is equivalent to S-equivalence for the associated element of  $\pi_0(\text{Diff}_0(S^1 \times S^3))$ . We show that every structure of a given type is S-equivalent to the trivial one if and only if the Schoenflies conjecture is true. We give conditions for interpolation of  $F|W$  systems and give sufficient conditions for  $F|W$  systems to be S-equivalent to the trivial one. Finally we state a *slice missing slice disc* problem related to  $F|W$  interpolation. Though they can be often stated in more generality, results in this section are given for  $S^1 \times S^3$  which we again denote by  $V$ . As before  $V_k$  will denote  $S^1 \times S^3 \#_k S^2 \times S^2$ ,  $k \in \mathbb{Z}_{\geq 0}$  and  $V_\infty$  will denote  $\mathbb{R} \times S^3 \#_\infty S^2 \times S^2$ .

**Definition 4.1.** The pseudo-isotopy  $f$  is *S-equivalent* to  $g$  if there exists a pseudo-isotopy  $\tilde{h} : \mathbb{R} \times S^3 \times I \rightarrow \mathbb{R} \times S^3 \times I$  such that  $\tilde{h}$  coincides with  $\tilde{f}$  near the  $-\infty$  end and with  $\tilde{g}$  near the  $+\infty$  end and we say that  $\tilde{f}$  *interpolates* to  $\tilde{g}$ .

Let  $p_t, q_t, t \in [0, 1]$  be 1-parameter families of  $V \times I \rightarrow [0, 1]$  such that  $p_0 = q_0$  is the standard projection and both  $p_1$  and  $q_1$  are non singular. Then  $p_t$  is *S-equivalent* to  $q_t$  if there exists a 1-parameter family  $\tilde{r}_t : \mathbb{R} \times S^3 \times I \rightarrow [0, 1], t \in [0, 1]$ , such that  $\tilde{r}_t$  coincides

with  $p_t$  (resp.  $q_t$ ) near the  $-\infty$  (resp.  $+\infty$ ) end,  $\tilde{r}_0$  is the standard projection and  $\tilde{r}_1$  is non singular.

The stable isotopies  $\Phi, \Phi'$  are S-equivalent if their lifts interpolate on  $V_\infty$ , i.e. if  $\Phi$  is defined on  $V_k$  and  $\Phi'$  on  $V_{k'}$ , then after identifying the  $-\infty$  (resp.  $+\infty$ ) end of  $V_\infty$  with the  $-\infty$  (resp.  $+\infty$ ) end of  $\tilde{V}_k$  (resp.  $\tilde{V}_{k'}$ ), there exists a stable isotopy on  $V_\infty$  that coincides with  $\tilde{\Phi}$  (resp.  $\tilde{\Phi}'$ ) near  $-\infty$  (resp.  $+\infty$ ).

The  $F|W$  systems  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}), (\mathcal{G}', \mathcal{R}', \mathcal{F}', \mathcal{W}')$  are S-equivalent if their lifts interpolate on  $V_\infty$ , i.e. there exists a  $F|W$  system on  $V_\infty$  that agrees with  $(\tilde{\mathcal{G}}, \tilde{\mathcal{R}}, \tilde{\mathcal{F}}, \tilde{\mathcal{W}})$  near  $-\infty$  and  $(\tilde{\mathcal{G}}', \tilde{\mathcal{R}}', \tilde{\mathcal{F}}', \tilde{\mathcal{W}}')$  near  $+\infty$ . Here we require that the manifolds obtained by surgering  $\mathcal{G}$  and  $\mathcal{G}'$  are diffeomorphic to  $\mathbb{R} \times S^3$ . If these systems are S-equivalent and  $\mathcal{G}' = \mathcal{G}, \mathcal{R}' = \mathcal{R}$  and  $\mathcal{F}' = \mathcal{F}$ , then we say that  $\mathcal{W}$  interpolates to  $\mathcal{W}'$ .

**Proposition 4.2.** *If for  $i = 0, 1, f_i$  is a pseudo-isotopy from id to  $\phi_i$ , then  $f_0$  is S-equivalent to  $f_1$  if and only if  $\phi_0$  is S-equivalent to  $\phi_1$ . S-equivalence on pseudo-isotopies on  $V \times I$  from id to elements of  $\text{Diff}_0(V)$  is an equivalence relation.*

*Proof.* Since S-equivalence is an equivalence relation on elements of  $\text{Diff}_0(V)$  and transitivity for S-equivalence of pseudo-isotopies is routine, the first assertion implies the second. The forward direction of the first assertion is immediate by definition.

We now show that if  $f$  is a pseudo-isotopy from id to  $\phi$  and  $\phi$  is S-equivalent to id then so is  $f$ . Let  $J = S^1 \times y \times I$ , some  $y \in S^3$ . Keeping  $S^1 \times S^3 \times 0$  fixed isotope  $f$  such  $f|N(J) = \text{id}$ . Let  $g = f|V \times I \setminus \text{int}(N(J))$  and consider  $\tilde{g} : \mathbb{R} \times B^3 \times I \rightarrow \mathbb{R} \times B^3 \times I$ . In what follows isotopies of  $g$  will be constant near the  $-\infty$  end as well as on  $\mathbb{R} \times \partial B^3 \times I \cup \mathbb{R} \times B^3 \times 0$ . Let  $B_t := t \times B^3 \subset \mathbb{R} \times B^3$  and  $\Delta_0^4 := \tilde{g}(B_0 \times I)$ . It follows from the proof of Theorem 9.11 [BG] that since  $\phi$  is S-equivalent to id,  $\Delta_0^4 \cap (\mathbb{R} \times B^3 \times 1)$  is ambiently properly isotopic to  $B_0 \times 1$  and hence we can assume by [Ce2] that  $\tilde{g}|N(B_0 \times 1) = \text{id}$ . Let  $s > 0$  be sufficiently large so that  $B_s \times I \cap \Delta_0^4 = \emptyset$ . Let  $\Delta^5$  be the closure of the region between  $B_s \times I$  and  $\Delta_0^4$ . Note that  $\Delta^5$  is connected and contractible. Also,  $\partial\Delta^5$  is diffeomorphic to  $S^4$ , again by [Ce2], since  $\partial\Delta^5$  is the union of two 4-balls glued along their boundaries. It follows [Sm2] that  $\Delta^5$  is diffeomorphic to a 5-ball. Essentially by uniqueness of regular neighborhoods we can ambiently isotope  $\Delta^4$ , and hence  $\tilde{g}$ , so that  $\Delta^4 = B_0 \times I := B_0^4$ . Since  $\tilde{g}|B_0^4 = \text{id}$  it follows that  $\tilde{g}|B_0^4 \in \text{Diff}_0(B_0^4 \text{ fix } \partial)$  and hence is pseudo-isotopic to id by [Br]. Therefore, there is an  $h : \mathbb{R} \times B^3 \times I \rightarrow \mathbb{R} \times B^3 \times I$  so that  $h|(-\infty, 0] \times B^3 \times I = \tilde{g}|(-\infty, 0] \times B^3 \times I$ ,  $h|[1, \infty) \times B^3 \times I = \text{id}$  and  $h|(\mathbb{R} \times B^3 \times 0) \cup \mathbb{R} \times (\partial B^3 \times I) = \text{id}$ . It follows that  $f$  is S-equivalent to id.  $\square$

**Proposition 4.3.** *The 1-parameter families  $q'_t$  and  $q_t$  are S-equivalent if and only if their corresponding pseudo-isotopies  $f', f$  are S-equivalent. S-equivalence of 1-parameter families is an equivalence relation.*

*Proof.* If  $q_t$  and  $q'_t$  are S-equivalent, then by restriction so are  $f$  and  $f'$ . The converse follows from the contractibility of the space of smooth functions to  $[0, 1]$ .  $\square$

**Proposition 4.4.** *The stable isotopies  $\Phi, \Phi'$  are S-equivalent if and only their corresponding transplantations  $f, f'$  are S-equivalent.*

Before the proof we give the following.

**Corollary 4.5.** *The  $F|W$  systems  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  and  $(\mathcal{G}', \mathcal{R}', \mathcal{F}', \mathcal{W}')$  are  $S$ -equivalent if and only if  $\phi := \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  and  $\phi' := \phi(\mathcal{G}', \mathcal{R}', \mathcal{F}', \mathcal{W}')$  are  $S$ -equivalent.*

*Proof.* If the  $F|W$  systems are  $S$ -equivalent, then the proof of Lemma 3.3 shows that they induce isotopic diffeomorphisms. Conversely, if  $\phi$  and  $\phi'$  are  $S$ -equivalent, then the  $F|W$  systems induce  $S$ -equivalent stable isotopies by the Proposition, hence the  $F|W$  systems are  $S$ -equivalent.  $\square$

*Proof of Proposition 4.4* The forward direction is immediate. To minimize notation we will consider the case that  $\Phi'$  is the trivial stable isotopy and hence  $f' = \text{id}$ , since general case is similar.

Suppose that  $(q_t, v_t)$  is a transplanted 1-parameter family arising from  $\Phi$ . It has a nested eye Cerf diagram only involving critical points of index-2 and 3 without 2/2 or 3/3 intersections, i.e. the corresponding 1-parameter family of handle structures on  $V \times I$  has no handle slides. Let  $(\tilde{q}_t, \tilde{v}_t)$  be the lift to  $\mathbb{R} \times S^3 \times I$ . Consider the  $\mathbb{R}$ -Cerf diagram on  $\mathbb{R} \times I \times I$ . Here a point  $(u, s, t)$  is labeled  $i$ , if  $\tilde{q}_t$  has a critical point  $\in u \times S^3 \times s$  of index- $i$ . The  $\mathbb{R}$ -Cerf diagram of  $(\tilde{q}_t, \tilde{v}_t)$  is a locally finite union of single eye components, only involving critical points of index-2 and 3, without 2/2 or 3/3 gradient intersections.

There is a generic 1-parameter family  $\tilde{p}_t : \mathbb{R} \times S^3 \times I \rightarrow [0, 1]$ ,  $t \in [0, 1]$  with glvf  $\tilde{w}_t$ , such that  $\tilde{p}_t|(\infty, -1] \times S^3 \times I = \tilde{q}_t$  and  $\tilde{p}_t|[1, \infty) \times S^3 \times I$  is the standard projection to  $[0, 1]$  with  $\tilde{w}_t$  the vertical vector field. Now and in the future  $\tilde{p}_t(\mathbb{R} \times S^3 \times i) = i$  for  $i$  close to 0 or 1;  $\tilde{p}_t$  is the standard projection to  $[0, 1]$  for  $t$  close to 0 and  $\tilde{p}_t$  is non singular for  $t$  close to 1. That such a  $(\tilde{p}_t, \tilde{w}_t)$  exists follows from the contractibility of smooth maps to  $[0, 1]$  and the hypothesis that  $f$  is  $S$ -equivalent to  $\text{id}$ . Also all modifications of  $\tilde{p}_t$  will be compactly supported, hence will always respectively agree with  $\tilde{q}_t$  and  $\text{id}$  near the negative and positive ends of  $\mathbb{R} \times S^3 \times I$ . To complete the proof it suffices to show that  $(\tilde{p}_t, \tilde{w}_t)$  can be modified so that the components of the  $\mathbb{R}$ -Cerf diagram are eyes with edges labeled 2 and 3 with independent births and deaths and without gradient intersections of type 2/2 or 3/3. The methods of §2 and §3 then show that after a compactly supported modification of  $\tilde{p}_t$  that the resulting proper stable isotopy is an interpolation of  $\Phi$  to the trivial stable isotopy.

We show that the proof of Proposition 3, Chapter VI [HW] extends to our setting. Since  $\tilde{p}_t$  is generic and for  $t \in [0, 1]$  agrees with either  $\tilde{q}_t$  or the standard projection off of a uniform compact set, all the components of the  $\mathbb{R}$ -Cerf diagram are compact, only finitely many are involved with 2/2 or 3/3 gradient intersections, or have critical points not of index-2 or 3 or have more than one birth or death, or have non independent births or deaths. Let  $C$  denote the union of these components. Since the proof establishing the first paragraph of p. 214 [HW] only requires that  $C$  be compact we can assume that the births and deaths of  $\tilde{p}_t$  are independent and the non birth/death critical points are of index-2 or 3. In a similar manner we can assume that conclusions of Step 1 and Steps 2 of the proof of Proposition 3 hold for  $\tilde{p}_t$ . Note that since  $\dim(V) = 4$ , the 3/3 gradient intersections are traded for 2/2 intersections. We continue to denote by  $C$  the components of the  $\mathbb{R}$ -Cerf diagram involved with 2/2 or 3/3 gradient intersections.

We now adapt Step 3 of the proof including its notation and terminology. Since the 2/2 intersections are supported in the compact set  $C$ , the argument of §6, Chapter V of [HW] shows that the 2/2 gradient intersections correspond to a word  $x \in K_2[Z[\pi_1(\mathbb{R} \times S^3 \times I)]]$  which is a subgroup of the Steinberg group  $\text{St}(\mathbb{Z}[1]) = \text{St}(\mathbb{Z})$ . Since  $W_2$  of the trivial group



equals 1 [HW] there is a word  $w \in W(\pm 1) \subset St(\mathbb{Z})$  such that  $x \cdot w = 1$  in  $St(r_1, \mathbb{Z})$  for  $r_1$  sufficiently large. (See p. 9-10 [HW] for definitions of these algebraic objects. Note that multiplicative group notation is used.) If  $r_0$  is the number of edges of  $C$  labeled 2 we can assume that  $r_0 \leq r_1$ . By modifying  $\tilde{p}_t$  to add  $r_1 - r_0$  trivial eye components we can assume that  $r_1 = r_0$ . By Lemma 2.7 Chapter IV [HW], we can modify  $\tilde{p}_t$  to add a new graphic representing the word  $w$  and then using (0.1) V [HW] to join the graphics to create a new one, still denoted  $C$ , whose word corresponding to the 2/2 gradient intersections is now  $x \cdot w$ . By the proof of Theorem 1.1, Chapter II [HW]  $p_t$  can be homotoped to eliminate these 2/2 gradient intersections via a homotopy that is supported on a compact region of  $\mathbb{R} \times S^3 \times I$ . No 3/3 gradient intersections are created in the above process.

Note that both after the births and before the deaths there is geometric  $\delta_{ij}$  intersection between the ascending spheres of the 2-handles and the descending spheres of the 3-handles. The 3/2 gradient intersections preserve the  $\delta_{ij}$  pairing algebraically, so if there are no 2/2 or 3/3 gradient intersections, edges that are born together must die together. These facts together with the fact that all the births and deaths are independent enable us to modify  $(\tilde{p}_t, \tilde{w}_t)$ , with compact support, so that the usual Cerf diagram is of the nested eye type, though infinitely many of the components of the  $\mathbb{R}$ -Cerf diagram may map to a single component of the usual diagram.  $\square$

Putting this all together we have the following

**Theorem 4.6.** *The following are equivalent*

- i) *The Schoenflies conjecture is true,*
- ii) *every  $\phi \in \text{Diff}_0(S^1 \times S^3)$  is S-equivalent to id,*
- iii) *every pseudo-isotopy on  $S^1 \times S^3 \times I$  from id is S-equivalent to id,*
- iv) *every  $F|W$  system on  $S^1 \times S^3$  is S-equivalent to the trivial system,*
- v) *every stable isotopy from the id on  $S^1 \times S^3$  is S-equivalent to id.*  $\square$

**Definition 4.7.** Define an abelian group structure on  $F|W$  systems on  $V$  whose elements are S-equivalence classes and whose addition is induced by the bijection with S-equivalence classes of  $\text{Diff}_0(V)$ .

We now describe interpolation operations on  $F|W$  systems.

**Lemma 4.8.** (*Disjoint Replacement*) *Let  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  and  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}')$  be  $F|W$  systems on  $V$ . Suppose that  $\partial\mathcal{W} = \partial\mathcal{W}'$  and  $\text{int}(\mathcal{W}) \cap \text{int}(\mathcal{W}') = \emptyset$ . Then the two systems are S-equivalent.*

*Proof.* Replace all the elements of  $\tilde{\mathcal{W}}$  near the  $+\infty$  end of  $V_\infty$  by corresponding elements of  $\tilde{\mathcal{W}}'$ .  $\square$

**Remarks 4.9.** i) Often a weaker version of this condition suffices, e.g.  $\partial\mathcal{W} = \partial\mathcal{W}'$  and  $\text{int}(\mathcal{W}) \cap \text{int}(\mathcal{W}') \neq \emptyset$  but after swapping  $\tilde{\mathcal{W}}$  discs by  $\tilde{\mathcal{W}}'$  discs near the  $+\infty$  end of  $V_\infty$  the resulting system of Whitney discs in  $V_\infty$  is embedded.

ii) Examples arise when  $\mathcal{W}'$  is obtained from  $\mathcal{W}$  by introducing local knotting and linking to its components.

iii) A variant where  $R_1 \cap (\mathcal{W} \cup \mathcal{F}), G_1 \cap (\mathcal{W} \cup \mathcal{F})$  are a union of intervals,  $|\mathcal{G}| = 1$ ,  $\partial\mathcal{W} = \partial\mathcal{W}'$  and  $\text{int}(\mathcal{W}) \cap \text{int}(\mathcal{W}') = \emptyset$  was given by Quinn [Qu] §4.5. It does not require passing to a cover and the operation does not change the isotopy class of the pseudo-isotopy. Quinn

attributes this operation in higher dimensions to Igusa. Question: Does *disjoint replacement* ever change the pseudo-isotopy class arising from  $\mathcal{F}|\mathcal{W}$ ?

Geometrically dual spheres disjoint from the finger and Whitney discs provide another useful tool.

**Lemma 4.10.** (*Dual Sphere Lemma*) *Let  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  be an  $F|W$  system supported on  $V_k$ . Suppose there exist pairwise disjoint embedded spheres  $H_{i_1}, \dots, H_{i_m}$  such that  $i_1, \dots, i_m$  are distinct elements of  $\{1, 2, \dots, k\}$  and*

$$i) H_{i_j} \cap (\mathcal{R} \cup \mathcal{G}) = H_{i_j} \cap (R_{i_j} \cup G_{i_j}) = 1 \text{ and}$$

$$ii) H_{i_j} \cap (\mathcal{F} \cup \mathcal{W}) = \emptyset \text{ all } i_j,$$

*then  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is  $S$ -equivalent to an explicit  $F|W$  system supported on some  $V_{k-m}$ . In particular if  $m = k$ , then this is the trivial system and  $\phi$  is  $S$ -equivalent to id.*

*Proof.* By reordering the  $S^2 \times S_i^2$  factors we can assume that  $i_j = j$ . Let  $\widetilde{S^2 \times S_i^2}, i \in \mathbb{Z}$  denote the preimages of the  $S^2 \times S_i^2$  factors in  $\widetilde{V}_k$  with the covering translation  $\rho$  shifting everything  $k$  units. We can assume that  $\mathcal{R}$  is in arm hand finger form.

If possible, reorder so that  $H_1 \cap G_1 \neq \emptyset$ , otherwise proceed as in four paragraphs below. If  $|G_1 \cap \mathcal{R}| = 1$ , then replace  $H_1$  by a translate of  $G_1$  and begin again. Observe that  $H_1$  has trivial normal bundle since  $[H_1] = [R_1] \in H_2(V_k)$  and is embedded. Modify  $\tilde{\mathcal{F}}|\tilde{\mathcal{W}}$  as follows. First, isotope  $\tilde{\mathcal{R}}$  to  $\tilde{\mathcal{R}}_1$  by doing Whitney moves using exactly all  $f \in \tilde{\mathcal{F}}$  such that  $\partial f \cap \tilde{G}_j \neq \emptyset$  where  $j = 1$  modulo  $k$  and  $j \geq 1$ . Denote by  $\hat{\mathcal{F}} \subset \tilde{\mathcal{F}}$  these  $f$ 's and  $\tilde{\mathcal{F}}_1 := \tilde{\mathcal{F}} \setminus \hat{\mathcal{F}}$ . Define  $\tilde{\mathcal{W}} = \{w \in \tilde{\mathcal{W}} | \partial w \cap G_j \neq \emptyset \text{ where } j = 1 \text{ modulo } k \text{ and } j \geq 1\}$  and  $\tilde{\mathcal{W}}' := \tilde{\mathcal{W}} \setminus \tilde{\mathcal{W}}$ .

The isotopy from  $\tilde{\mathcal{R}}$  to  $\tilde{\mathcal{R}}_1$  is supported very close to  $\hat{\mathcal{F}}$ , in particular so that  $\tilde{\mathcal{R}}_1$  remains disjoint from the  $\tilde{H}_i$ 's. If  $w \in \tilde{\mathcal{W}}'$  and  $w \cap \hat{\mathcal{F}} \neq \emptyset$ , then under isotopy extension  $w$  gets moved to  $w_1$ . Let  $\mathcal{W}'_1 = \tilde{\mathcal{W}}'$  with these  $w$ 's replaced by their  $w_1$ 's.

This creates two types of problems for  $(\tilde{\mathcal{G}}, \tilde{\mathcal{R}}_1, \tilde{\mathcal{F}}_1, \tilde{\mathcal{W}}'_1)$  being an  $F|W$  system. If  $f \in \hat{\mathcal{F}}$  and  $w \in \tilde{\mathcal{W}}'$  with  $|\text{int } f \cap w| = p$ , then  $w_1$  will have  $2p$  new intersections with some  $\tilde{G}_j$  while if  $|w \cap \partial f| = p$ , then  $w_1$  will have  $p$  additional new intersections with some  $\tilde{G}_j$ . In both cases,  $j \geq 1$  and  $j = 1$  modulo  $k$ . Now modify the  $w_1$ 's by tubing off with copies of the component of  $\tilde{H}_1$  that intersects  $\tilde{G}_j$ . Let  $\tilde{\mathcal{W}}_1$  denote the modified  $\tilde{\mathcal{W}}'_1$ .

Do all the modifications  $\rho$ -equivariantly. This means that if  $f \in \hat{\mathcal{F}}$ , then the Whitney move associated to  $\rho(f)$  is  $\rho$  of the Whitney move associated to  $f$ . Similarly, if  $w \in \tilde{\mathcal{W}}'$  and  $w \cap \hat{\mathcal{F}} \neq \emptyset$ , then  $(\rho(w))_1 \cap N(\hat{\mathcal{F}}) = \rho(w_1) \cap N(\hat{\mathcal{F}})$ . Here the isotopy of  $\tilde{\mathcal{R}}$  to  $\tilde{\mathcal{R}}_1$  is supported in  $N(\hat{\mathcal{F}})$ . Also the tubings are done  $\rho$ -equivariantly as well. It follows that if  $(\tilde{\mathcal{G}}, \tilde{\mathcal{R}}_1, \tilde{\mathcal{F}}_1, \tilde{\mathcal{W}}_1)$  is the new system, then the restriction to the  $+\infty$  end projects to a  $F|W$  system  $(\mathcal{G}, \mathcal{R}_1, \mathcal{F}_1, \mathcal{W}_1)$  on  $V_k$  such that  $|\mathcal{R}_1 \cap G_1| = 1$  and  $H_1, \dots, H_m$  satisfy the same properties as before with this  $F|W$  system. Now replace  $H_1$  by a translate  $H'_1$  of  $G_1$ .

By induction on  $|\cup H_i \cap \mathcal{G}|$  we now assume that for all  $i, H_i \cap \mathcal{G} = \emptyset$ . Abuse notation by denoting the new  $F|W$  system arising from the construction of the previous paragraphs by  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$ . Now modify  $(\tilde{\mathcal{G}}, \tilde{\mathcal{R}}, \tilde{\mathcal{F}}, \tilde{\mathcal{W}})$  as follows. For  $i \geq 1$  and  $i = j$  modulo  $k$ , where  $1 \leq j \leq m$ , obtain  $\tilde{R}'_i$  from  $\tilde{R}_i$  by doing Whitney moves using all  $f \in \tilde{\mathcal{F}}$  such that  $\partial f \cap \tilde{R}_i \neq \emptyset$ . Again, the Whitney moves are done very close to these  $f$ 's and  $\rho$ -equivariantly. Let  $\tilde{\mathcal{R}}_1$  denote the modified  $\tilde{\mathcal{R}}$ . Let  $\hat{\mathcal{F}} \subset \tilde{\mathcal{F}}$  denote the union of discs used in these moves

and  $\tilde{\mathcal{F}}_1 = \tilde{\mathcal{F}} \setminus \hat{\mathcal{F}}$ . Let  $\hat{W} = \{w \in \tilde{\mathcal{W}} \mid w \cap \tilde{R}_i \neq \emptyset \text{ for } i \geq 1 \text{ and } i = j \text{ modulo } k, \text{ where } 1 \leq j \leq m\}$ . Let  $\tilde{\mathcal{W}}' = \tilde{\mathcal{W}} \setminus \hat{W}$ .

The problem with  $(\tilde{\mathcal{G}}, \tilde{\mathcal{R}}_1, \tilde{\mathcal{F}}_1, \tilde{\mathcal{W}}')$  is that if  $w \in \tilde{\mathcal{W}}'$ , then  $w \cap \tilde{\mathcal{R}}_1$  may have intersections. Indeed if  $f \in \hat{F}$ , then each point of  $\text{int}(f) \cap w$  will give rise to two such intersections and each point of  $\partial f \cap w$  will give rise to one. Modify  $\tilde{\mathcal{W}}'$  to  $\tilde{\mathcal{W}}_1$  by  $\rho$ -equivariantly tubing off each  $\tilde{\mathcal{W}}' \cap \tilde{\mathcal{R}}_1$  intersection with a copy of some  $\tilde{H}_i$ . Project the  $+\infty$  end of  $(\tilde{\mathcal{G}}, \tilde{\mathcal{R}}_1, \tilde{\mathcal{F}}_1, \tilde{\mathcal{W}}_1)$  to  $V_k$  to obtain an  $F|W$  system satisfying the conclusions of the lemma.

If  $m = k$ , then the resulting  $F|W$  system will have  $\mathcal{F} = \emptyset$  and hence the corresponding  $\phi$  is S-equivalent to id. Also, the interpolation is done explicitly and algorithmically. Indeed, the final  $F|W$  system on  $V_k$  can be explicitly constructed directly in  $V_k$ .  $\square$

**Definition 4.11.** A component *component*  $\mathcal{C} = (\mathcal{G}_C, \mathcal{R}_C, \mathcal{F}_C, \mathcal{W}_C)$  of  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  consists of those elements lying in a connected component of  $\mathcal{G} \cup \mathcal{R} \cup \mathcal{F} \cup \mathcal{W}$ . It is supported in  $V_{k-m}$  where  $m = |\mathcal{G} \setminus \mathcal{G}_C|$  and is obtained from  $V_k$  by surgering the components of  $\mathcal{G} \setminus \mathcal{G}_C$ . *Deletion* of  $\mathcal{C}$  is the  $F|W$  system obtained by removing  $\mathcal{C}$  from  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  and surgering  $\mathcal{G}_C$ . Note that it is supported on  $V_m$ . We say that  $\mathcal{C}$  is *S-trivial* if the induced map  $\phi(\mathcal{C})$  is S-equivalent to id. The  $F|W$  system is *connected* if it has one component.

**Remarks 4.12.** i) If each component of  $\tilde{\mathcal{C}}$  is compact, then  $\mathcal{C}$  is S-trivial.

ii) If  $h \in \text{Diff}_0(B^4 \text{ fix } \partial)$ , then  $h$  is pseudo-isotopic to the identity so by [HW] and [Qu] it arises from an  $F|W$  system on  $B^4 \#_k S^2 \times S^2$  which can be viewed as an S-trivial  $F|W$  system on  $S^1 \times S^3$ .

iii) Concatenation can be viewed as the operation of addition of unions of components.

**Lemma 4.13.** *If the  $F|W$  system  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is the disjoint union of components  $\cup_{i=1}^m (\mathcal{G}_i, \mathcal{R}_i, \mathcal{F}_i, \mathcal{W}_i)$ , then  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}) = \sum_{i=1}^m \phi(\mathcal{G}_i, \mathcal{R}_i, \mathcal{F}_i, \mathcal{W}_i)$ .*

*Proof.* By disentangling in  $V_\infty$  we will show that  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is S-equivalent to the concatenation of its components. We give the proof for  $m = 2$ , the general case follows by induction. Suppose that  $|\mathcal{G}_i| = k_i$ . View  $\mathcal{R}$  in arm hand finger form and let  $L$  denote an unknotted  $S^1 \times S^2$  which separates  $\mathcal{G}_1 \cup \mathcal{R}_1 \cup \mathcal{F}_1 \cup \mathcal{R}_1^{\text{std}}$  from  $\mathcal{G}_2 \cup \mathcal{R}_2 \cup \mathcal{F}_2 \cup \mathcal{R}_2^{\text{std}}$ . This means that the closed complementary regions are copies of  $S^1 \times B^3 \#_{k_i} S^2 \times S^2$ ,  $i = 1, 2$  respectively denoted  $V'_{k_1}, V'_{k_2}$ . Note that  $\mathcal{W}_1$  is disjoint from  $\mathcal{G}_2 \cup \mathcal{R}_2 \cup \mathcal{F}_2$  which deformation expands to a space  $X_2 \subset V'_{k_2}$  which is  $N(\mathcal{G}_2 \cup \mathcal{R}_2^{\text{std}})$  union finitely many 1-handles. Using isotopy extension we can assume that  $\mathcal{W}_1 \cap X_2 = \emptyset$ . Let  $\mathcal{W}'_1$  denote the result of isotoping  $\mathcal{W}_1$  into  $V'_{k_1}$  via an isotopy fixing  $\mathcal{G}_1 \cup \mathcal{R}_1 \cup \mathcal{R}_1^{\text{std}}$  pointwise. The track of this isotopy may cross  $X_2$ . In a similar manner construct  $\mathcal{W}'_2$ .

We now show that  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is S-equivalent to the concatenation of  $(\mathcal{G}_1, \mathcal{R}_1, \mathcal{F}_1, \mathcal{W}'_1)$  and  $(\mathcal{G}_2, \mathcal{R}_2, \mathcal{F}_2, \mathcal{W}'_2)$ . Consider  $\tilde{V}_k$  where  $k = k_1 + k_2$ . Let  $\widetilde{S^2 \times S^2}_i$ ,  $i \in \mathbb{Z}$  denote the preimages of the  $S^2 \times S^2$  factors in  $V_k$  with the covering translation  $\rho$  shifting everything  $k$  units. Order the factors so that when  $i = j$  modulo  $k$ , where  $1 \leq j \leq k_1$  (resp.  $k_1 + 1 \leq j \leq k$ ), then these factors lift from  $V'_{k_1}$  (resp.  $V'_{k_2}$ ). Now let  $\tilde{\mathcal{W}}_i^1$  be a Whitney system for  $\tilde{\mathcal{G}}_i \cup \tilde{R}_i$  that coincides with  $\tilde{\mathcal{W}}_i$  near  $-\infty$  and  $\tilde{\mathcal{W}}'_i$  near  $+\infty$  and  $\tilde{\mathcal{W}}_i^1 \cap \tilde{X}_j = \emptyset$  where  $i \neq j$ . Such systems exist because we can  $\rho$ -equivariantly isotope  $\tilde{\mathcal{W}}_i$  to  $\tilde{\mathcal{W}}'_i$  only near the  $+\infty$  end, so don't have unwanted intersections between moved and unmoved components of  $\tilde{\mathcal{W}}_i^1$ .

To complete the proof we modify  $\tilde{W}_1^1$  to eliminate the finite set  $\tilde{W}_1^1 \cap \tilde{W}_2^1$ . First isotope  $\tilde{W}_1^1$  off of  $\tilde{W}_2^1$  to obtain  $\tilde{W}_1^2$  at the cost of creating twice as many  $\tilde{W}_1^2 \cap \tilde{R}_2$  intersections. Now  $\tilde{R}_2$  has pairwise disjoint geometrically dual spheres that are disjoint from  $\tilde{G}_2 \cup \tilde{W}_2^1 \cup \tilde{W}_1^2 \cup \tilde{G}_1 \cup \tilde{R}_1$  by doing Whitney moves to  $\tilde{G}_2$  using  $\tilde{W}_2^1$  and isotoping slightly. Finally create  $\tilde{W}_1^3$  by tubing  $\tilde{W}_1^2$  to copies of the dual spheres, one for each point of  $\tilde{W}_1^2 \cap \tilde{R}_2$ .  $\square$

**Corollary 4.14.** (*Deletion Lemma*) *If  $\mathcal{F}'|\mathcal{W}'$  is the subsystem of  $\mathcal{F}|\mathcal{W}$  with all  $S$ -trivial components deleted, then  $\phi(\mathcal{F}|\mathcal{W})$  is  $S$ -equivalent to  $\phi(\mathcal{F}'|\mathcal{W}')$ .*  $\square$

**Definition 4.15.** Let  $\mathcal{G}_1 \subset \mathcal{G}, \mathcal{R}_1 \subset \mathcal{R}$  be such that  $\mathcal{G}_1 \cup \mathcal{R}_1$  is a union of components of  $\mathcal{G} \cup \mathcal{R}$ . We say that  $(\mathcal{G}, \mathcal{R}', \mathcal{F}', \mathcal{W}')$  is obtained by *contracting*  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  along  $\mathcal{G}_1, \mathcal{R}_1$ , if  $\mathcal{R}'$  is obtained by replacing  $\mathcal{R}_1$  by  $\mathcal{R}_1^{\text{std}}$ ;  $\mathcal{F}' = \{f \in \mathcal{F} | \partial f \cap \mathcal{R}_1 = \emptyset\}$ ;  $\mathcal{W}'' = \{w \in \mathcal{W} | \partial w \cap \mathcal{R}_1 = \emptyset\}$  and  $\mathcal{W}' = \mathcal{W}''$  modified as follows. If  $w \in \mathcal{W}''$  and  $\text{int}(w) \cap \mathcal{R}_1^{\text{std}} \neq \emptyset$ , then modify by tubing each intersection of  $w \cap R_j^{\text{std}}$  with a copy of  $G_j$ .

**Remark 4.16.** Since we can assume that  $\mathcal{R}$  is in AHF form with respect to  $\mathcal{F}$ ,  $\mathcal{R}_1^{\text{std}} \cap (\mathcal{R} \setminus \mathcal{R}_1) = \emptyset$ .

**Lemma 4.17.** (*Contraction Lemma*) *If  $(\mathcal{G}, \mathcal{R}', \mathcal{F}', \mathcal{W}')$  is obtained from  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  by contracting along those components of  $\mathcal{G} \cup \mathcal{R}$  whose induced maps into  $\pi_1(V_k)$  are trivial, then  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is  $S$ -equivalent to  $\phi(\mathcal{G}, \mathcal{R}', \mathcal{F}', \mathcal{W}')$ . In particular, if all components of  $\mathcal{G} \cup \mathcal{R}$  are  $\pi_1$ -inessential, then  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is  $S$ -equivalent to id.*

*Proof.* Let  $U$  be a neighborhood of the  $+\infty$  end of  $\tilde{V}_k$  that is disjoint from some neighborhood of the  $-\infty$  end. Replace all the lifts of the red spheres contained in all the  $\pi_1$ -inessential components of  $\mathcal{G} \cup \mathcal{R}$  which intersect  $U$  by the corresponding standard red spheres. Obtain the new  $\tilde{\mathcal{F}}$  by deleting those elements whose boundaries intersected the red spheres that were replaced. Obtain the new  $\tilde{\mathcal{W}}$  by also deleting those elements whose boundaries intersected the replaced red spheres. Modify those remaining discs  $\in \tilde{\mathcal{W}}$  that intersect the new red spheres by tubing off intersections using copies of their dual green spheres.  $\square$

**Definition 4.18.** We say that  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  has *fingers monotonically pointing up* (resp. *down*) if the  $S^2 \times S^2$  factors can be ordered in  $V_k$  so that in  $\tilde{V}_k$ ,  $\tilde{R}_i \cap \tilde{G}_j \neq \emptyset$  implies  $j \geq i$  (resp.  $j \leq i$ ).

**Proposition 4.19.** *If  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  has either fingers monotonically pointing up or down, then  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is  $S$ -equivalent to id.*

*Proof.* Start with  $\mathcal{R}$  in arm hand finger form and the  $S^2 \times S^2$  factors of  $\tilde{V}_k$  ordered as usual with the covering translation  $\rho$  shifting  $\widetilde{S^2 \times S^2}_i$  to  $\widetilde{S^2 \times S^2}_{i+k}$ . It suffices to consider the case that all the fingers point down. Replace  $\tilde{R}_i, i \geq 0$  by  $\tilde{R}_i^{\text{std}}$  and eliminate those elements of  $\tilde{F} \cup \tilde{W}$  which intersect  $\tilde{R}_i, i \geq 0$ . The cost is that some of the remaining elements of  $\tilde{W}$  may intersect  $\tilde{R}_i^{\text{std}}, i \geq 0$ , however there are only finitely many such intersections. Tube off each of intersections with  $\tilde{R}_i^{\text{std}}$  with a copy of  $\tilde{G}_i$ . Since  $(\tilde{\mathcal{G}}, \tilde{\mathcal{R}}, \tilde{\mathcal{F}}, \tilde{\mathcal{W}})$  interpolates to  $(\tilde{\mathcal{G}}, \tilde{\mathcal{R}}^{\text{std}}, \emptyset, \emptyset)$  the result follows.  $\square$

**Remark 4.20.** This argument more generally shows the following. Given  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  as in Notation 3.5, define a graph  $G \subset S^1 \times D^2$  as follows, where  $S^1 = [0, k]/\sim$ . The vertices are  $\{i \times 0 | i \in k\}$ . To each finger  $f$  from  $R_i$  to  $G_j$ , construct an embedded directed edge

from  $i \times 0$  to  $j \times 0$  which winds  $\omega(f)$  about the  $S^1$ . If no component of  $G$  contains both a cycle representing a positive element of  $\pi_1$  and a cycle representing a negative element, then  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is S-equivalent to id.

The next lemma shows that if  $\mathcal{F}$  and  $\mathcal{W}$  have common sets of dual spheres then they are interpolable.

**Definition 4.21.** A set of pairwise disjoint embedded 2-spheres  $\mathcal{N}_f = \{N_1, \dots, N_n\}$  with trivial normal bundles is said to be *dual* to  $\mathcal{F} = \{F_1, \dots, F_n\}$  if  $\mathcal{N}_f \cap \mathcal{G} \cup \mathcal{R} = \emptyset$  and  $|N_i \cap F_j| = \delta_{ij}$ . In a similar manner we define the notion of dual spheres to  $\mathcal{W}$ .

**Lemma 4.22.** (*Whitney duals exist*) *Given the  $F|W$  system  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$ , there are dual spheres  $\mathcal{N}_f$  and  $\mathcal{N}_w$  to  $\mathcal{F}$  and  $\mathcal{W}$ .*

*Proof.* We argue as in [FQ]. Given  $F_i$ , let  $D_i$  be a small 2-sphere near  $F_i \cap R_i$  that intersects  $F_i$  once and  $R_i$  twice. For each  $i$ , tube  $D_i$  with two copies of the dual sphere  $R'_i$  to  $R_i$ , where  $R'_i$  is as in Figure 1, to eliminate these points and thereby construct  $N_i$  and hence  $\mathcal{N}_f$ . In a similar manner construct  $\mathcal{N}_w$ .  $\square$

**Remarks 4.23.** We can also construct  $\mathcal{N}_f$  or  $\mathcal{N}_w$  by starting with spheres that each intersect  $\mathcal{G}$  twice. In general  $\mathcal{N}_f$  is not obviously equal  $\mathcal{N}_g$  and  $\mathcal{N}_f \cap \mathcal{N}_w \neq \emptyset$ , which essentially is the cause of our difficulties.

**Lemma 4.24.** *Let  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  be a  $F|W$  system such that the boundary germs of  $\mathcal{F}$  coincide with that of  $\mathcal{W}$ .*

- i) If  $\mathcal{N}_f = \mathcal{N}_w$ , then  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is S-equivalent to id.*
- ii) If  $\mathcal{W}'$  is another set of Whitney discs with  $\mathcal{N}_w = \mathcal{N}_{w'}$ , then  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is S-equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}')$ .*

*Proof.* To prove i) we show that  $\tilde{\mathcal{W}}$  interpolates to  $\tilde{\mathcal{F}}$ . In  $\tilde{V}_k$  attempt to construct a interpolating system  $\tilde{\mathcal{W}}'$  by using  $\tilde{\mathcal{W}}$  near the  $-\infty$  end and  $\tilde{\mathcal{F}}$  near the  $+\infty$  end. While each component of  $\tilde{\mathcal{W}}'$  is embedded a component coming from  $\tilde{\mathcal{W}}$  may intersect one from  $\tilde{\mathcal{F}}$ . Since  $\tilde{\mathcal{N}}_f$  is a common system of dual spheres to both  $\tilde{\mathcal{F}}$  and  $\tilde{\mathcal{W}}$  we can use copies of components of  $\tilde{\mathcal{N}}_f$  to tube away these intersections. In a similar manner prove ii) by showing that  $\mathcal{W}$  interpolates to  $\mathcal{W}'$ .  $\square$

**Question 4.25.** (*Second Test Case*) *Let  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  be such that  $\{\mathcal{G}\} = G_1$ ,  $R_1$  has exactly two fingers respectively of winding  $\pm 1$  and each Whitney disc coincides with a finger disc in a neighborhood of its boundary. Is  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  S-equivalent to id?*

**Problem 4.26.** (*Slice missing slice disc problem*). *The knot  $K \subset S^1 \times S^2 = \partial S^1 \times B^3$  shown in Figure 2 bounds two obvious ribbon discs  $D_1$  and  $D_2$  such that the simple closed curve  $\alpha \subset S^1 \times S^2 \setminus K$  (resp.  $\beta$ ) slices in  $S^1 \times B^3$  with a slice disc disjoint from  $D_1$  (resp.  $D_2$ ). Is it true that for any smooth disc  $D$  bounded by  $K$ , one of  $\alpha$  or  $\beta$  slices in the complement of  $D$ ?*

## 5. TWISTED WHITNEY DISCS

The goal of this section and the next is to show that a  $F|W$  system interpolates to one whose finger and Whitney discs coincide along neighborhoods of their boundaries, i.e. their

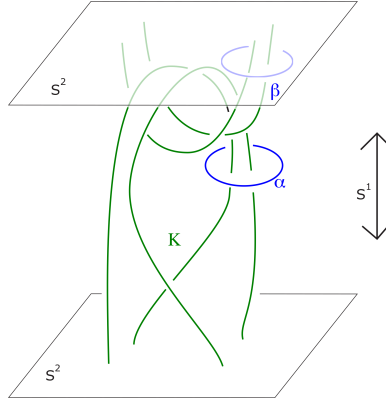


FIGURE 2. Slice Missing Slice Disc

*boundary germs* coincide. Such a  $F|W$  system is called *boundary germ coinciding*. To do this we first arrange for their boundaries to coincide and then get neighborhoods of their boundaries to coincide. The proof of the latter uses the main result of this section, Lemma 5.4 which subject to a certain technical condition that is satisfied by first passing to a finite cover, asserts that given any system  $\mathcal{F}$  of Whitney discs for  $\mathcal{G}$  and  $\mathcal{R}$  in  $S^1 \times S^3 \times \#_k S^2 \times S^2$ , there exists another system  $\mathcal{F}'$  such that  $\mathcal{F}'$  has prescribed twisting relative to that of  $\mathcal{F}$  and  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}')$  is S-equivalent to the identity.

We now define the twisting of one Whitney disc relative to another, by putting a neighborhood of the boundary of one into a normal form relative to that of the other.

**Definition 5.1.** Let  $w_0$  and  $w_1$  be Whitney discs for the oriented, possibly disconnected surfaces  $G$  and  $R$  in the oriented 4-manifold  $M$  such that  $\partial w_0 = \partial w_1$ . Let  $x$  and  $y$  denote the points of  $w_i \cap G \cap R$  where  $x$  is the point of  $-1$  intersection. Define  $\beta = w_i \cap G$  and  $\alpha = w_i \cap R$  with  $\alpha$  (resp.  $\beta$ ) oriented from  $x$  to  $y$  (resp.  $y$  to  $x$ ). See Figure 3 a) which shows a 3-dimensional slice of  $M$  that contains  $w_0$ . We assume that the orientation of  $G$  is given by  $(\epsilon_1, \epsilon_2)$  and at  $y$ ,  $R$  is oriented by  $(\epsilon_3, \epsilon_4)$ . After an isotopy of  $w_1$  we can assume that it coincides with  $w_0$  near both  $x$  and  $y$ . It follows that after a further isotopy, a neighborhood of  $\partial w_1$  rotates  $p \in \mathbb{Z}$  times along  $\alpha$  and  $q \in \mathbb{Z}$  times along  $\beta$ . Here  $q$  (resp.  $p$ ) is the number of full right hand twists about  $\beta$  (resp.  $\alpha$ ), the former using the convention of Figure 3 b). Figure 3 c) shows the projection to the 3-dimensional slice of a neighborhood of the boundary of a  $(3, 1)$ -twisted disc to the  $(x, y, z)$  plane. We call  $w_1$  a  $(p, q)$ -twisted Whitney disc rel  $w_0$  and we define  $\text{tw}(w_1, w_0) = (p, q)$ .

**Lemma 5.2.** *If  $w_0, w_1$ , and  $w_2$  are Whitney discs with  $\partial w_2 = \partial w_1 = \partial w_0$ , then  $\text{tw}(w_2, w_0) = \text{tw}(w_2, w_1) + \text{tw}(w_1, w_0)$ . Also  $\text{tw}(w_0, w_1) = -\text{tw}(w_1, w_0)$ .  $\square$*

**Lemma 5.3.** *If  $w_1 \subset M$  is a  $(p, q)$ -twisted Whitney disc rel  $w_0$  and  $M$  has trivial second Stiefel - Whitney class, then  $p + q$  is even.*

*Proof.* The normal bundle of the Whitney disc  $w_1$  for surfaces  $R$  and  $G$  has a framing which when restricted to  $\partial D$  has, where applicable, one vector tangent to  $R$  and transverse to  $G$  and the other tangent to  $G$  and transverse to  $R$ . On the other hand, starting with  $w_0$  perform the boundary twisting operation, e.g. see [E] P. 216, to obtain an embedded pre-Whitney

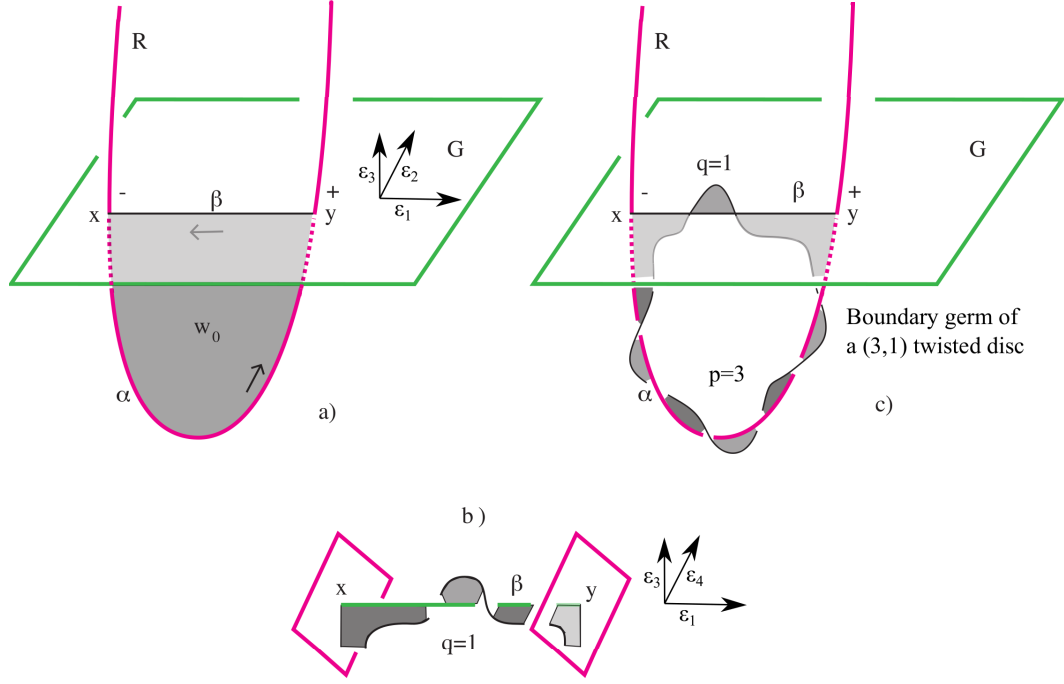


FIGURE 3. The boundary germ of a twisted Whitney disc

disc  $E$  whose boundary has a neighborhood that coincides with that of  $w_1$  but whose framing differs from that of a genuine Whitney disc by  $p + q \bmod 2$ . By gluing  $E$  to  $w_1$  along their boundaries and smoothing near the gluing we obtain a smoothly immersed 2-sphere in  $M$  with normal bundle of Euler class equal to  $p + q \bmod 2$ . Since  $w_2(M) = 0$  it follows that  $p + q = 0 \bmod 2$ .  $\square$

The following is the main result of this section. As before  $V_k$  denotes  $S^1 \times S^3 \times \#_k S^2 \times S^2$  and  $\mathcal{G}$  and  $\mathcal{R}$  denote sets of algebraically dual embedded 2-spheres as in Definition 3.1.

**Lemma 5.4.** *Let  $\mathcal{F} = \{f_1, \dots, f_n\}$  be a complete set of Whitney discs for  $\mathcal{G} = \{G_1, \dots, G_k\}$  and  $\mathcal{R} = \{R_1, \dots, R_k\}$  in  $V_k$  such that the winding of any hand from any  $R_i$  to  $G_i$  is equal to zero. Let  $((p_1, q_1), \dots, (p_n, q_n)) \in \mathbb{Z} \oplus \mathbb{Z}$  be such that for all  $i$ ,  $p_i + q_i$  is even. Then there exists a system of Whitney discs  $\mathcal{F}' = \{f'_1, \dots, f'_n\}$  such that  $\partial \mathcal{F} = \partial \mathcal{F}'$ ,  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}')$  is  $S$ -equivalent to the identity and for  $i = 1, 2, \dots, n$ ,  $\text{tw}(f'_i, f_i) = (p_i, q_i)$ .*

**Remark 5.5.** Let  $z_r \in G_r \cap R_r$  denote the point disjoint from all the  $f_i$ 's. The winding hypothesis implies that if there is a Whitney disc  $f_i$  between  $G_r$  and  $R_r$ , then the element of  $\pi_1(V_k)$  corresponding to a loop starting at  $z_r$  that follows  $R_r$  to a point of  $G_r \cap R_r \cap f_i$ , then follows  $G_r$  back to  $z_r$  is homotopically trivial.

*Proof. Step 1:* The lemma holds when  $(p_i, q_i) = (0, 0)$  for all  $i > 1$  and either  $(p_1, q_1) = (0, \pm 2)$  or  $(\pm 2, 0)$ .

*Proof of Step 1.* Suppose that  $f_1$  cancels points of  $G_r \cap R_s$ . We consider the  $(p_1, q_1) = (2, 0)$  case as the other cases are similar. Use the boundary twisting operation [E]) to obtain an embedded  $(2, 0)$  twisted pre-Whitney disc  $E_0$  which fails to be a genuine Whitney disc

because its framing is off by two and it intersects  $R_s$  twice. Correct the framing by replacing a small disc with one of self intersection  $\pm 1$  (see [FQ] p. 14) and then push the intersection off the  $\alpha$  boundary to obtain the embedded disc  $E_1$  which has the correct framing but  $|E_1 \cap R_s| = 4$ .

Let  $R'_s$  be a geometrically dual sphere to  $R_s$  disjoint from  $(\mathcal{G} \cup (R \setminus R_s) \cup \mathcal{F} \cup E_1)$ . Such a sphere can be obtained from  $G_s$  by doing Whitney moves to  $G_s$  using the components of  $\mathcal{F}$  that intersect  $G_s$  and then isotoping slightly. Here we assume that  $E_1$  is constructed to lie very close to  $f_1$ . Next eliminate the four  $\text{int}(E_1) \cap R_s$  intersections by tubing  $E_1$  to four parallel copies of  $R'_s$  along arcs in  $R_s$  disjoint from  $\mathcal{F}$ . Thus we obtain a Whitney disc  $f'_1$  with  $\text{tw}(f'_1, f_1) = (2, 0)$  and  $f'_1 \cap (\mathcal{F} \setminus f_1) = \emptyset$ . Let  $\mathcal{F}_1 = \{f'_1, f_2, \dots, f_n\}$ . Now  $(\tilde{G}, \tilde{R}, \tilde{F}, \tilde{F}_1) \subset \tilde{V}_k$  interpolates to  $(\tilde{G}, \tilde{R}, \tilde{F}, \tilde{F})$  by replacing lifts of  $f'_1$  with lifts of  $f_1$  near the  $+\infty$  end of  $\tilde{V}_k$ . This uses the fact that a lift of  $f'_1$  is disjoint from  $\tilde{F}$  except for the single component that has the same boundary. It follows that  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}_1)$  is  $S$ -equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F})$  which is the class of the identity.  $\square$

*Step 2:* If the Lemma holds for  $(p_1, q_1) = (p, q)$  and  $(p_i, q_i) = (0, 0)$  for  $i > 1$ , then the Lemma holds for  $(p_1, q_1) = (p \pm 2, q)$  and  $(p, q \pm 2)$  with  $(p_i, q_i) = (0, 0)$  for  $i > 1$ .

*Proof of Step 2.* Let  $\mathcal{F}'$  be a set of Whitney discs for which the hypothesis of Step 2 holds. Now apply Step 1 to  $\mathcal{F}'$  to obtain  $\mathcal{F}_1$ . By Lemma 5.2  $\mathcal{F}_1$  satisfies the twisting conclusion of Step 2 relative to  $\mathcal{F}$ . Also by Lemma 3.15,  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}_1) = \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}', \mathcal{F}_1) \circ \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}')$  and so is  $S$ -equivalent to the identity.  $\square$

*Step 3:* The lemma holds when  $(p_1, q_1) = (1, 1)$ ,  $(p_i, q_i) = (0, 0)$  for  $i > 1$  and  $f_1 \cap G_r \cap R_s \neq \emptyset$  where  $r \neq s$ .

*Proof of Step 3.* The proof is a modification of the proof of Step 1. Construct a framed, embedded pre-Whitney disc  $E_0$  with  $\text{tw}(E_0, f_1) = (1, 1)$  disjoint from  $G'_r \cup R'_s$  by starting with  $f_1$  and then doing boundary twisting operations by twisting once about  $f_1 \cap G_r$  and once about  $f_1 \cap R_s$ . If necessary, correct the framing by first replacing a small embedded disc by one with self intersection  $\pm 1$  and then making the resulting pre-Whitney disc embedded by pushing the self intersection off the  $f_1 \cap R_s$  component. Except for its intersections with  $G_r$  and  $R_s$  the resulting disc  $E_1$  is a genuine Whitney disc.

Construct geometric dual spheres  $R'_s$  (resp.  $G'_r$ ) from  $G_s$  (resp.  $R_r$ ) disjoint from  $(\mathcal{G} \cup (\mathcal{R} \setminus R_s) \cup \mathcal{F} \cup E_1)$  (resp.  $(\mathcal{G} \setminus G_r) \cup \mathcal{R} \cup \mathcal{F} \cup E_1$ ). Construct  $f'_1$  by tubing off  $E_1 \cap (R_s \cup G_r)$  using copies of  $R'_s$  and  $G'_r$  and tubes that avoid the discs of  $\mathcal{F}$ . The argument of Step 1 shows that if  $\mathcal{F}' = \{f'_1, f_2, \dots, f_n\}$ , then  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}')$  is  $S$ -equivalent to the identity.  $\square$

*Step 4:* The lemma holds when  $(p_1, q_1) = (1, 1)$ ,  $(p_i, q_i) = (0, 0)$  for  $i > 1$  and  $f_1 \cap G_1 \cap R_1 \neq \emptyset$ .

*Proof of Step 4.* As in the previous steps it suffices to show that there exists a Whitney disc  $f'_1$  with  $\partial f'_1 = \partial f_1$  and  $\text{tw}(f'_1, f_1) = (1, 1)$  such that for  $i > 1$ ,  $f'_1 \cap f_i = \emptyset$  and in  $\tilde{V}_k$ , if  $\tilde{f}'_1$  and  $\tilde{f}_1$  are lifts of  $f'_1$  and  $f_1$ , then  $\tilde{f}'_1 \cap \tilde{f}_1 \neq \emptyset$  if and only if  $\partial \tilde{f}'_1 = \partial \tilde{f}_1$ .

To simplify notation denote  $G_1$  and  $R_1$  by  $G$  and  $R$ . By Lemma 3.11 we can assume that  $\mathcal{R}$  is in finger hand form with respect to  $\mathcal{G}$ . See Figure 1. Since the winding of the hand of  $R$  containing  $f_1$  equals 0, it follows that if this hand has  $s \geq 1$  fingers, then there are  $s$  alternative Whitney discs  $a_1, \dots, a_s$  as in Figure 4 a), where the  $s = 2$  case is shown. Note



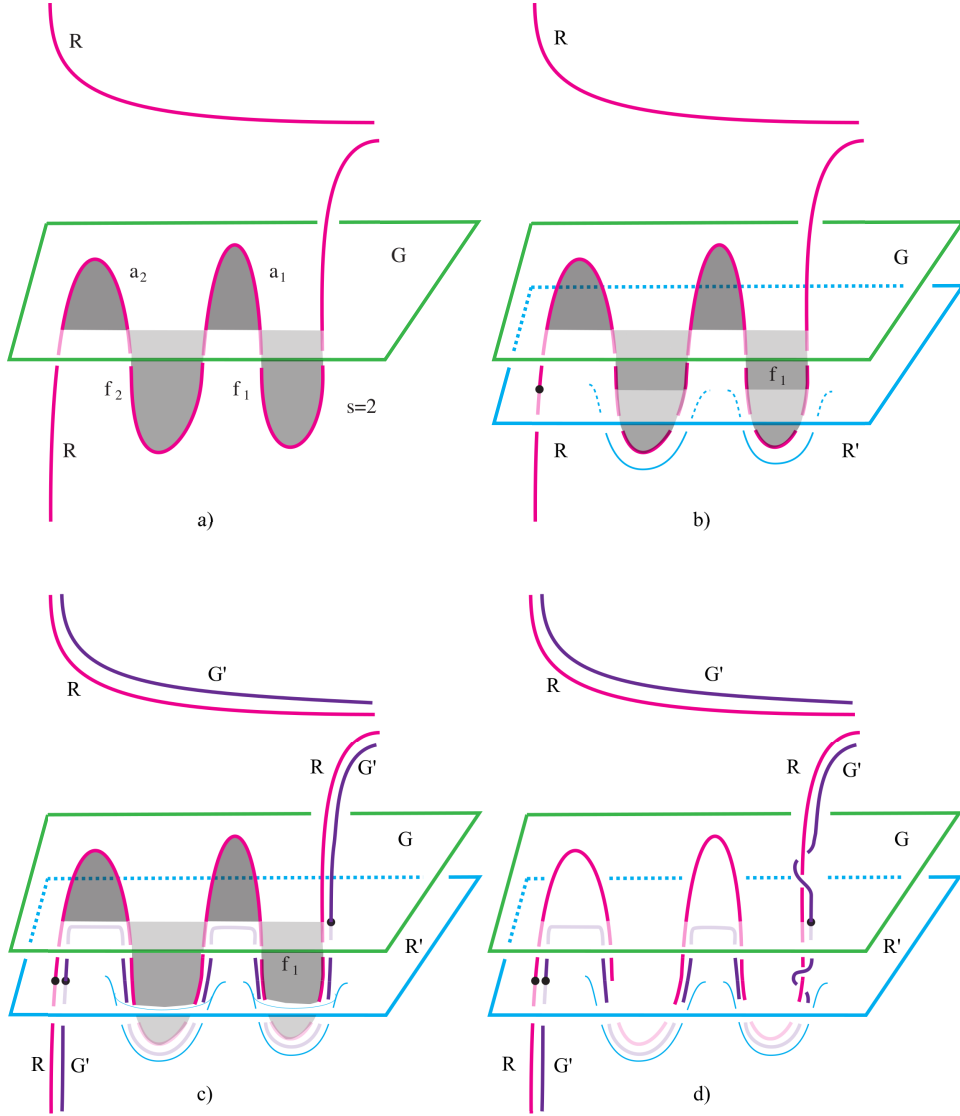


FIGURE 4. Constructing dual spheres

that by choosing the  $a_i$ 's appropriately we can assume that the hand's fingers and alternate discs appear as in the figure, in particular  $f_1$  is the first in the indicated sequence of  $a_i$ 's and  $f_j$ 's.

Now construct the geometrically dual sphere  $R'$  for  $R$ . As before construct  $R'$  by applying Whitney moves to  $G$  using the applicable discs of  $\mathcal{F}$ . After a slight isotopy  $R' \cap (G \cup \mathcal{F}) = \emptyset$  and  $|R' \cap \mathcal{R}| = |R' \cap R| = 1$ . See Figure 4 b). Next construct the geometrically dual sphere  $G'$  for  $G$  by first doing Whitney moves to  $R$  along  $a_1, \dots, a_s$  as well as Whitney moves to all the relevant discs in  $\mathcal{F} \setminus \{f_1, \dots, f_s\}$ . Construct  $G'$  to lie close to  $R$  and these discs, in particular  $G' \cap R'$  is very close to  $R \cap R'$ . After a slight isotopy we can assume that  $|G' \cap \mathcal{G}| = |G' \cap G| = 1$  and  $G' \cap (\mathcal{R} \cup \mathcal{F}) = \emptyset$ . See Figure 4 c).

Next isotope  $G'$  so that its intersection with a 3-ball containing  $f_1$  has a full right hand twist about  $R$  near  $f_1$  which is compensated by a full left hand twist as in Figure 4 d). This

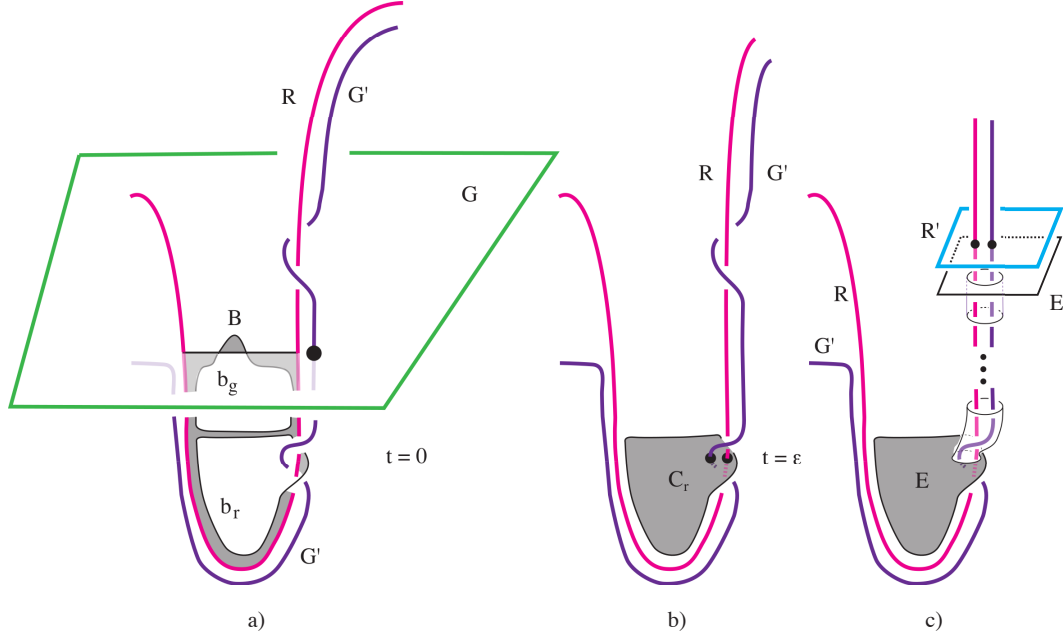


FIGURE 5. Constructing a (1,1)-twisted Whitney disc

twist gets undone when moving both in the past and in the future and creates an intersection with  $f_1$ .

We now construct  $f'_1$ . Let  $A$  be an annulus having (1,1)-twisting near  $\partial f_1$  with one component of  $\partial A$  equal to  $\partial f_1$ . By construction,  $A \cap G' = \emptyset$ . Next add an untwisted band to  $A$  to create  $B$  as in Figure 5 a). Again  $B \cap G' = \emptyset$ . Let  $b_r$  and  $b_g$  denote the components of  $\partial B$  that link around  $R$  and  $G$ . Construct an embedded correctly framed pre-Whitney disc  $C$  by capping off these components with discs  $C_r$  and  $C_g$ .  $C_r$  consists of an annulus starting at  $b_r$  that moves directly into the future which is then capped off by a disc in a horizontal time slice. So if Figure 5 a) shows a  $t = 0$  slice, then Figure 5 b) shows the  $t = \epsilon$  slice and the spanning disc. The disc  $C_g$  bounded by  $b_g$  will have an excess intersection point with  $G$ . Since  $1 + 1 = 2$ , for  $C$  to have the correct framing the construction of  $C_g$  might require first creating a self intersection and then pushing off to create two extra intersections with  $G$ . Note that the unique intersection point of  $C \cap G'$  is next to the unique transverse intersection point of  $C$  with  $R$  as indicated in Figure 5 b). Next using a tube that follows parallel arcs in  $R$  and  $G'$ , tube off these two points using a copy of  $R'$ . See Figure 5 c). Let  $E$  denote the resulting framed embedded pre-Whitney disc. Note that  $E \cap G' = \emptyset$ . Finally tube off the one (or three) excess points of  $E \cap G$  with copies of  $G'$  to construct the desired Whitney disc  $f'_1$ .  $\square$

*Step 5:* The lemma holds when  $(p_i, q_i) = (0, 0)$  for  $i \neq 1$ .

*Proof of Step 5* First assume that  $p_1$  is even, and hence by parity so is  $q_1$ . Then there exists a sequence  $(0, 0) = (p^1, q^1), (2, 0) = (p^2, q^2), \dots, (p^r, q^r) = (p_1, q_1)$  so that for  $1 \leq m \leq r - 1$ ,  $(p^{m+1}, q^{m+1}) - (p^m, q^m) = (\pm 2, 0)$  or  $(0, \pm 2)$ . Let  $\mathcal{F} = \mathcal{F}^1, \mathcal{F}^2, \dots, \mathcal{F}^r$  be complete systems of Whitney discs for  $\mathcal{R}$  and  $\mathcal{G}$  such that for  $1 \leq m \leq r$ ,  $\mathcal{F}^m = \{f^m, f_2, \dots, f_n\}$  where  $f^1 = f_1$ ,  $\text{tw}(f^2, f^1) = (2, 0)$ , for  $2 \leq m \leq r - 1$ ,  $\text{tw}(f^{m+1}, f^m) = ((p^{m+1}, q^{m+1}) - (p^m, q^m))$ , and for

all  $i$ ,  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}^i, \mathcal{F}^{i+1})$  is  $S$ -equivalent to the identity. The existence of  $\mathcal{F}^2$  follows by Step 1 and the existence of  $\mathcal{F}^i$ ,  $i > 2$  follows from Step 2. By Lemma 5.2  $\text{tw}(f^r, f_1) = (p_1, q_1)$  and by factorization  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}^r) = \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}^{r-1}, \mathcal{F}^r) \circ \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}^{r-2}, \mathcal{F}^{r-1}) \circ \cdots \circ \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}^1, \mathcal{F}^2)$ . Since each of the factors is  $S$ -equivalent to the id, the result follows.

Next assume that  $p_1$  is odd. Then there exists a sequence  $(0, 0) = (p^1, q^1), (1, 1) = (p^2, q^2), \dots, (p^r, q^r) = (p_1, q_1)$  so that for  $2 \leq m \leq r-1$ ,  $(p^{m+1}, q^{m+1}) - (p^m, q^m) = (\pm 2, 0)$  or  $(0, \pm 2)$ . Argue as above, except use one of Steps 3 or 4 in place of Step 1.  $\square$

*Step 6:* The lemma holds when  $(p_i, q_i) = 0$  for  $i \neq s$ .  $\square$

*Proof of Step 6:* This follows from Step 5 after reordering the elements of  $\mathcal{F}$ .  $\square$

*Step 7: General Case*

*Proof of Step 7:* Through repeated uses of Step 6, Lemma 5.2 and factorization, inductively construct  $\mathcal{F}_0 = \mathcal{F}, \mathcal{F}_1, \dots, \mathcal{F}_n = \mathcal{F}'$  so that  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}_i)$  is  $S$ -equivalent to id and if  $\mathcal{F}_i = (f_1^i, \dots, f_n^i)$ , then  $\text{tw}(f_j^i, f_j) = (p_j, q_j)$  if  $j \leq i$  and  $(0, 0)$  otherwise.  $\square$

## 6. GERMS OF FINGER AND WHITNEY DISCS

The goal of this section is to prove the following:

**Proposition 6.1.** *Let  $\phi \in \text{Diff}_0(S^1 \times S^3)$ , then  $\phi$  is  $S$ -equivalent to  $\psi \in \text{Diff}_0(S^1 \times S^3)$  arising from a boundary germ coinciding  $F|W$  system.*

**Lemma 6.2.** *Given  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  there exists  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}')$  such that  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}')$  is  $S$ -equivalent to id and  $\mathcal{W} \cap \mathcal{G} = \mathcal{W}' \cap \mathcal{G}$ .*

*Proof.* View  $\mathcal{G}$  and  $\mathcal{R}$  in AHF form. See Figure 1 which shows the restriction of  $\mathcal{R} \cup \mathcal{G}$  to the  $i$ 'th  $S^2 \times S^2$  factor, the finger discs which intersect  $G_i$  as well as spheres  $R'_i$  and  $G'_i$  respectively geometrically dual to  $R_i$  and  $G_i$ . Letting  $\mathcal{R}' = \cup_{R_i \in \mathcal{R}} R'_i$ , note that  $\mathcal{R}' \cap \mathcal{G} = \emptyset$  and is geometrically dual to  $\mathcal{R}$ . Let  $w \in \mathcal{W}$ . If  $w \cap G_i \cap R_j \neq \emptyset$  and  $\beta_w := w \cap G_i$ , then since  $\partial w$  is homotopically trivial in  $V_k$ ,  $\partial \beta_w$  lies in a single hand  $H_w \subset R_j$ . Let  $\gamma_w$  be an embedded loop of the form  $\beta_w \cup \alpha_w$  where  $\alpha_w \subset H_w$  and has interior disjoint from the *finger arcs*  $\mathcal{F} \cap \mathcal{R}$ . The loops  $\cup_{w \in \mathcal{W}} \gamma_w$  can be chosen to be pairwise disjoint. Each  $\gamma_w$  bounds an immersed disc  $D_w$  contained in it's  $S^2 \times S^2$  factor whose interior is disjoint from  $\mathcal{G} \cup \mathcal{R}'$ . These discs may have intersections and self intersections and possibly  $\text{int}(D_w) \cap \mathcal{R} \neq \emptyset$ . At the cost of creating additional intersections with  $\mathcal{R}$ , these discs, which continue to be called  $D_w$ 's, can be made embedded and pairwise disjoint. By boundary twisting near the  $\alpha_w$ 's, thereby creating further intersections with  $\mathcal{R}$ , these discs can be correctly framed. They may fail to be Whitney discs only because the interior of the  $D_w$ 's intersect  $\mathcal{R}$  transversely. Finally eliminate these intersections by tubing with parallel copies of components  $\mathcal{R}'$  using tubes that follow paths in  $\mathcal{R}$ . Let  $\mathcal{W}'$  denote the resulting collection of Whitney discs. Since  $\mathcal{R}' \cap (\mathcal{W}' \cup \mathcal{F}) = \emptyset$ , it follows from Lemma ?? that  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}')$  is  $S$ -equivalent to id.  $\square$

**Lemma 6.3.** *Given  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  such that  $\mathcal{F} \cap \mathcal{G} = \mathcal{W} \cap \mathcal{G}$  there exists  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}'')$  such that  $\partial \mathcal{W}'' = \partial \mathcal{W}$  and  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}'')$  is  $S$ -equivalent to the identity.*

*Proof.* View  $\mathcal{G}$  in AHF form with respect to  $\mathcal{F}$  and let  $\mathcal{G}'$  denote the geometrically dual spheres to  $\mathcal{G}$  as above disjoint from  $\mathcal{R}$ . If  $w \in \mathcal{W}$ , then since  $w \cap \mathcal{G} \subset \mathcal{F} \cap \mathcal{G}$ , i.e. is a

finger arc,  $\partial w$  bounds an immersed disc whose interior is disjoint from  $\mathcal{R} \cap \mathcal{G}'$ . As in the previous lemma these discs can be modified to construct a family of Whitney discs  $\mathcal{W}''$  with  $\partial \mathcal{W}'' = \partial \mathcal{W}$  and  $\mathcal{W}'' \cap \mathcal{G}' = \emptyset$  and hence  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}'')$  is  $S$ -equivalent to the identity.  $\square$

**Lemma 6.4.** *Given  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  there exists  $(\mathcal{G}, \mathcal{R}, \mathcal{F}_1, \mathcal{W}_1)$  such that  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is  $S$ -equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}_1, \mathcal{W}_1)$  and  $\partial \mathcal{F}_1 = \partial \mathcal{W}_1$ .*

*Proof.* Apply Lemma 6.2 to find  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}')$  such that  $\mathcal{W}' \cap \mathcal{G} = \mathcal{W} \cap \mathcal{G}$  and  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}')$  is  $S$ -equivalent to the id. By Lemma 3.15,  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is isotopic to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{W}', \mathcal{W}) \circ \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}')$  and hence  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is  $S$ -equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{W}', \mathcal{W})$ .

Now apply Lemma 6.3 to find  $(\mathcal{G}, \mathcal{R}, \mathcal{W}', \mathcal{W}'')$  such that  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{W}', \mathcal{W}'')$  is  $S$ -equivalent to id and  $\partial \mathcal{W} = \partial \mathcal{W}''$ . Again by factorization  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{W}', \mathcal{W})$  is  $S$ -equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{W}'', \mathcal{W}) \circ \phi(\mathcal{G}, \mathcal{R}, \mathcal{W}', \mathcal{W}'')$ . It follows that  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is  $S$ -equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{W}'', \mathcal{W})$  where  $\partial \mathcal{W} = \partial \mathcal{W}''$ .  $\square$

*Proof of Proposition 6.1.* Let  $\phi \in \text{Diff}_0(S^1 \times S^3)$ . Suppose that it is represented by  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  which is supported on  $V_k$ . Staying within the  $S$ -equivalence class of  $\phi$ , we can additionally assume by Lemma 6.4 that  $\partial \mathcal{F} = \partial \mathcal{W}$ . By passing to a finite sheeted cover of  $S^1 \times S^3$  and hence  $V_k$ , an operation preserving  $S$ -equivalence, we can assume that every hand from an  $R_i$  to its  $G_i$  has winding 0. Indeed, any finite cover of degree greater than the maximal winding over all hands from an  $R_i$  to its  $G_i$  suffices. Order the elements  $(f_1, \dots, f_n)$  of  $\mathcal{F}$  and  $(w_1, \dots, w_n)$  of  $\mathcal{W}$  so that for each  $i$ ,  $\partial f_i = \partial w_i$ . Let  $(p_i, q_i) = \text{tw}(w_i, f_i)$ . Now apply Lemma 5.4 to find  $\mathcal{F}'$  so that  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}')$  is  $S$ -equivalent to the identity and  $\text{tw}(\mathcal{F}', \mathcal{F}) = ((p_1, q_1), \dots, (p_n, q_n))$ . It follows that  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is  $S$ -equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}', \mathcal{W}) \circ \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}')$  and hence to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}', \mathcal{W})$ . By Lemma 5.2,  $\text{tw}(\mathcal{F}', \mathcal{W}) = \text{tw}(\mathcal{F}', \mathcal{F}) + \text{tw}(\mathcal{F}, \mathcal{W}) = \text{tw}(\mathcal{F}', \mathcal{F}) - \text{tw}(\mathcal{W}, \mathcal{F}) = ((0, 0), \dots, (0, 0))$ .  $\square$

It follows from Theorem 4.6 and Proposition 6.1

**Theorem 6.5.** *The Schoenflies conjecture is true if and only if every boundary germ coinciding  $F|W$  system interpolates to the trivial  $F|W$  system.*

## 7. HOMOTOPIC WHITNEY AND FINGER DISCS

The following is the main result of this section.

**Proposition 7.1.** *If  $\phi \in \text{Diff}_0(S^1 \times S^3)$ , then up to  $S$ -equivalence  $\phi$  is represented by an  $F|W$  system on some  $V_k$  such that if  $\mathcal{F} = (f_1, \dots, f_m)$  and  $\mathcal{W} = (w_1, \dots, w_m)$ , then for every  $i$  the boundary germ of  $w_i$  coincides with that of  $f_i$  and  $w_i$  is homotopic to  $f_i$  via a homotopy fixing  $N(\partial w_i)$  pointwise and supported in  $V_k \setminus \mathcal{G} \cup \mathcal{R}$ .*

**Remarks 7.2.** Using Proposition 6.1 we will start with an  $F|W$  system  $\mathcal{F} = (f_1, \dots, f_m)$  and  $\mathcal{W} = (w_1, \dots, w_m)$  satisfying the boundary germ conclusion. We will assume that  $\mathcal{R}$  is in AHF form with respect to  $\mathcal{F}$ . Also for  $i \neq j$ ,  $R_i \cap G_j$  is contained in a single hand, since this can be achieved by passing to a finite cover, an  $S$ -equivalence preserving operation. In addition, we will assume that for all  $i$ ,  $R_i \cap G_{i+1} \neq \emptyset$ . If necessary, achieve this by adding extra hands with equal finger and Whitney discs. Here indices in  $\mathbb{Z}$  are modulo  $k$ .

**Notation 7.3.** We continue to use notation as in 3.7. Define  $X := N(\mathcal{G} \cup \mathcal{R})$ ,  $Y := V_k - \text{int } X$ ,  $U_0 = \cup_{i=1}^k S^2 \times S_i^2$  and  $Z := V_k \setminus \cup_{i=1}^k \text{int}(S^2 \times S_i^2)$ . Let  $A_{i,j}$  denote the arm that goes from  $R_i$  to  $G_j$ , provided one exists and define  $A_{i,j}^Z := Z \cap A_{i,j}$ .

**Lemma 7.4.** *By passing to another finite cover and isotoping the  $A_{i,j}^Z$ 's we can assume that  $\pi(A_{i,j}^Z) \subset$  exactly one of  $(i, j)$  or  $(j, i)$ ,  $\text{diam } \pi(A_{i,j}^Z) < k/16$  and for each  $w_p \in \mathcal{W}$ ,  $\text{diam } \pi(w_p \cap Z) < k/16$ .  $\square$*

**Notation 7.5.** Let  $I_{i,j}$  denote the short subinterval of  $S^1$  bounded by  $i, j$ . In what follows this interval will usually have length  $\leq k/4$ . Define a directed graph  $G$  whose vertices are the  $\Sigma_i$ 's and whose edges  $E := \{e_{i,j}\}$  are the  $A_{i,j}$ 's with  $i \neq j$ , where  $e_{i,j}$  points from  $\Sigma_i$  to  $\Sigma_j$ . Let  $C$  denote the cycle formed by  $e_{0,1}, \dots, e_{k-1,k}$ . For  $e_{i,j} \in E$ , let  $H_{i,j} \subset Z$  denote a 1-handle with attaching discs in  $\Sigma_i$  and  $\Sigma_j$  such that  $\pi(H_{i,j}) \subset I_{i,j}$ . Assume that the  $H_{i,j}$ 's are pairwise disjoint. Let  $U_1 = U_0 \cup_{i,j} H_{i,j}$ .

**Lemma 7.6.** *The  $A_{i,j}^Z$ 's can be naturally isotoped into the  $H_{i,j}$ 's after which  $U_1$  deformation retracts to  $X \cup N(\mathcal{F} \cup \mathcal{A})$ , where  $\mathcal{A} = \{a_1, \dots, a_n\}$  are the auxiliary discs.  $\square$*

*Proof.* Thicken the finger discs to 2-handles and expand them to first fill the fingers, then the hands and then the arms. Finally add  $N(\mathcal{A})$ . The result is isotopic to  $N(\mathcal{G}^{\text{std}} \cup \mathcal{R}^{\text{std}})$  union 1-handles, one for every arm from  $R_i$  to  $G_j$ ,  $i \neq j$ . Note that an arm from an  $R_i^{\text{std}}$  to  $G_j^{\text{std}}$  together with its finger and auxiliary discs gets absorbed into  $N(\mathcal{G}^{\text{std}} \cup \mathcal{R}^{\text{std}})$ . After the  $A_{i,j}^Z$ 's have been naturally isotoped to lie in the  $H_{i,j}$ 's we see that  $U_1$  deformation retracts to  $X \cup N(\mathcal{F} \cup \mathcal{A})$ .  $\square$

**Lemma 7.7.** *There exists a system of discs  $\mathcal{D} := \cup_{e \in E \setminus C} D_e := \{D_1, \dots, D_r\}$  called arm rest discs such that*

- i) The elements of  $\mathcal{D}$  are pairwise disjoint and properly embedded in  $Y$*
- ii)  $\mathcal{D} \cap (\mathcal{F} \cup \mathcal{A}) = \emptyset$*
- iii) If  $U = X \cup N(\mathcal{F} \cup \mathcal{A} \cup \mathcal{D})$ , then  $S^1 \times S^3 \setminus \text{int}(U)$  is isotopic to a vertical  $S^1 \times B^3$ .*

*Proof.* The arm rest discs will have the property that if  $D_i$  corresponds to  $e_{p,q}$ , then  $D_i$  runs over  $H_{p,q}$  exactly once and is disjoint from all the other  $H_{p,q}$ 's except those of the form  $H_{i,i+1}$ . Assuming conclusions i) and ii), it follows that when thickened to 2-handles the  $D_i$ 's cancel the  $H_{i,j}$ 's of  $U_1$  with  $j \neq i+1$  and hence the resulting  $U$  is isotopic to  $\cup_{i=1}^k S^2 \times S_i^2 \cup H_{i,i+1}$  and so its closed complement is isotopically a vertical  $S^1 \times B^3$ . We detail the construction of a special case, say  $D_1 := D_{e_{3,1}}$ , from which the general construction may be deduced. Our  $D_1$  is bounded by  $\alpha * \beta$  where  $\alpha \subset \partial N(A_{3,1})$  with initial point in  $N(R_3^{\text{std}})$  and final point in  $N(A_{3,1}) \cap N(G_1^{\text{std}})$ . Then  $\beta$  follows a path  $\subset \partial N(G_1^{\text{std}} \cup R_1^{\text{std}} \cup A_{1,2} \cup G_2^{\text{std}} \cup R_2^{\text{std}} \cup A_{2,3} \cup G_3^{\text{std}} \cup R_3^{\text{std}})$ .  $\square$

**Definition 7.8.** Let  $\mathcal{E} := \mathcal{F} \cup \mathcal{A} \cup \mathcal{D}$ . Fix orientations on the elements of  $\mathcal{E}$  and then induce orientations on the elements of  $\mathcal{W}$  from those of  $\mathcal{F}$  using the boundary germ condition.

**Lemma 7.9.** *If  $S$  closed oriented surface in  $Y$ , then  $[S] = 0 \in H_2(Y)$  if and only if for each  $E \in \mathcal{E}$ ,  $\langle S, E \rangle = 0$ .*

*Proof.* The forward direction follows from the fact that algebraic intersection number is a homological invariant. Conversely, if all the intersection numbers equal 0, then  $S$  is homologous to a surface disjoint from  $U$  and hence is homologically trivial, since  $H_2(S^1 \times B^3) = 0$ .  $\square$

**Definition 7.10.** For  $i \in \{1, \dots, k\}$ , let  $R'_i \subset S^2 \times S_i^2$  be a dual sphere to  $R_i$  as in Figure 1, i.e. is obtained by choosing a parallel copy of  $G_i^{\text{std}}$  that intersects  $R_i$  exactly once and  $R'_i \cap \mathcal{F} = \emptyset$ . Construct oriented pairwise disjoint *linking spheres* to the elements of  $\mathcal{E}$ , i.e.

to  $E \in \mathcal{E}$  we define an oriented embedded sphere  $T_E$  with trivial normal bundle such that  $T_E \cap (\mathcal{E} \cup \mathcal{G} \cup \mathcal{R}) = T_E \cap E$  is a single point of positive sign. We do this first on the finger and auxiliary discs. Let  $E$  be such a disc with say  $E \cap R_j \neq \emptyset$ . Let  $T'_E$  be a 2-sphere consisting of an annulus disjoint from  $R_j$  that intersects  $E$  once in its interior together with two discs that each intersect  $R_j$  once of opposite sign. These  $T'_E$ 's can be chosen to be oriented, pairwise disjoint and so that  $T'_E \cap \mathcal{E} = T'_E \cap E = 1$  positive point. To obtain  $T_E$  tube off the two intersections with parallel copies of  $R'_j$  where the tubes follow arcs in  $R_j$  [No]. This can be done maintaining pairwise disjointness and so that each  $T_E$  is disjoint from all the  $R'_j$ 's. Given an arm rest disc  $D$  that intersects  $H_{i,j}, j \neq i+1$ , construct the oriented linking sphere  $T_D \subset Z$  using a sphere that links  $H_{i,j}$ .

**Remark 7.11.** Note that each  $T_E$  can be constructed so that  $\text{diam}(\pi(T_E \cap Z)) < k/16$ .

**Lemma 7.12.** *i)  $H_2(Y)$  is freely generated by  $\{[T_f] | f \in \mathcal{F}\} \cup \{[T_a] | a \in \mathcal{A}\} \cup \{[T_D] | D \in \mathcal{D}\}$ .*

*ii)  $H_2(Y, \partial\mathcal{E})$  is freely generated by  $\{[E] | E \in \mathcal{E}\}$  and  $\{[T_f] | f \in \mathcal{F}\} \cup \{[T_a] | a \in \mathcal{A}\} \cup \{[T_D] | D \in \mathcal{D}\}$ .*  $\square$

**Lemma 7.13.** *If  $w_i \in \mathcal{W}$ , then the coefficient of the  $[T_{f_i}]$  term of  $[w_i] \in H_2(Y, \partial\mathcal{E})$  equals zero.*

*Proof.* If this coefficient was non zero, then  $w_i$  would have a framing inconsistent with that of a Whitney disc with the same boundary germ as  $f_i$ .  $\square$

**Lemma 7.14.** *If  $n_{p,q}$  denotes the coefficient of the  $[T_{f_q}]$  term of  $[w_p] \in H_2(Y, \partial\mathcal{E})$ , then  $n_{i,j} = -n_{j,i}$ .*

*Proof.* Since  $w_i \cap w_j = \emptyset$  it follows that  $0 = \langle [w_i], [w_j] \rangle = n_{i,j} + n_{j,i}$ . The latter equality follows since  $[w_i] = [f_i] + n_{i,j}T_{f_j}$  and  $[w_j] = [f_j] + n_{j,i}T_{f_i}$  plus other terms that do not contribute to intersection number.  $\square$

**Lemma 7.15.** *There exists a system  $\mathcal{F}' = (f'_1, \dots, f'_m)$  of Whitney discs with the same boundary germs as  $\mathcal{F}$  such that for each  $p$ ,  $[f'_p] = [w_p] \in H_2(Y, \partial\mathcal{E})$  and  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}')$  is  $S$ -equivalent to  $\text{id}$ . Finally, for each  $p$ ,  $\text{diam}(\pi(f'_p \cap Z)) \leq k/4$ .*

*Proof.* Construct  $f_p^1$  by first tubing  $f_p$  to  $n_{p,q}$  copies of  $T_{f_q}$ , when  $q < p$  and then taking the disjoint union with  $n_{p,q}$  parallel copies of  $T_{f_q}$  when  $p < q$ . The  $f_p^1$ 's can be constructed to be embedded, disjoint from each  $R'_j$ .

Construct  $f_1^2, f_2^2, \dots, f_m^2$  as follows. For  $q > p$  tube the disc component of  $f_p^1$  to its  $T_{f_q}$  components using tubes that follow arcs in  $f_q^1$  connecting oppositely oriented points of  $f_p^1 \cap f_q^1$ . This can be done so that the  $f_p^2$ 's are pairwise disjoint, embedded and disjoint from each  $R'_j$ .

Finally obtain  $f'_p$  by tubing  $f_p^2$  to  $\langle w_p, a_r \rangle$  parallel copies of  $T_{a_r}$  and  $\langle w_p, D_s \rangle$  parallel copies of  $T_{D_s}$ . This whole construction can be done so that the  $f'_p$ 's are pairwise disjoint, disjoint from each  $R'_j$  and each  $\text{diam}(\pi(f'_p \cap Z)) \leq k/4$ . By construction  $[f'_p] = [w_p]$  all  $p$ . Since for all  $j$ ,  $(\mathcal{F} \cup \mathcal{F}') \cap R'_j = \emptyset$  it follows that  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}')$  is  $S$ -equivalent to  $\text{id}$ .  $\square$

*Proof of Proposition 7.1.* By Proposition 6.1  $\phi$  is  $S$ -equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  where the boundary germs of  $\mathcal{F}$  and  $\mathcal{W}$  coincide. After lifting to finite cover we can assume that the conclusion of Lemma 7.4 holds. Now let  $\mathcal{F}'$  be as in Lemma 7.15. By factorization  $\phi$  is  $S$ -equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}', \mathcal{W}) \circ \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{F}')$  and hence  $\phi$  is  $S$ -equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}', \mathcal{W})$ .

Since the inclusion  $Y \rightarrow V_k$  is a  $\pi_1$ -isomorphism, it follows that  $\tilde{Y} = \tilde{V}_k \setminus N(\tilde{G} \cup \tilde{R})$ . Since  $\text{diam}(\pi(f'_p \cup w_p)) \cap Z \leq k/4$  it follows that  $f'_p$  is homologous to  $w_p$  by a chain disjoint from  $\pi^{-1}(j + 1/2)$  for some  $j \in \{1, 2, \dots, k\}$ . It follows if  $\tilde{f}'_p, \tilde{w}_p$  are lifts with common boundary germs, then they are homologous in  $H_2(\tilde{Y}, \partial\tilde{\mathcal{E}})$  and hence are homotopic rel partial in  $\tilde{Y}$ . Therefore, for all  $p$ ,  $f'_p$  is homotopic rel  $\partial$  to  $w_p$  in  $Y$ .  $\square$

## 8. HOMOTOPY IMPLIES CONCORDANCE

In [MM] Maggie Miller shows that in a 4-manifold whose fundamental group contains no 2-torsion, homotopic 2-spheres are concordant provided one of them has a dual 2-sphere and under suitable hypothesis this holds more generally. Here we use §4.1 [MM] to show that homotopic  $F|W$  systems are S-equivalent to ones with  $\mathcal{W}$  concordant to  $\mathcal{F}$ .

**Definition 8.1.** The  $F|W$  system  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  on  $V_k$  is  *$\partial$ -germ coinciding* if  $\partial\mathcal{W}$  coincides with  $\partial\mathcal{F}$  near its boundary. We say that  $\mathcal{W}$  is *homotopic* to  $\mathcal{F}$  if they are  $\partial$ -germ coinciding and if  $\mathcal{W}$  can be homotoped to  $\mathcal{F}$  rel  $\partial$  by a homotopy supported in  $V_k \setminus (\mathcal{G} \cup \mathcal{R})$ . We say that  $\mathcal{W}$  is *standardly concordant* to  $\mathcal{F}$  if it is obtained by starting with  $\mathcal{F} \times [0, 1]$ , attaching cancelling 3-dimensional 1- and 2-handles to  $\mathcal{F} \times 1$  and then reimbedding the 2-handles to obtain an immersed  $\mathcal{F} \times [0, 1]$  whose  $\mathcal{F} \times 1 := \mathcal{W}_1$  is embedded. This also describes a standard concordance as a critical level embedding. The reimbedded 2-handles are required to coincide with the original ones near their attaching regions. Finally  $\mathcal{W}$  is obtained from  $\mathcal{W}_1$  by isotoping slightly to regain the  $\partial$ -germ coinciding condition.

Each  $w \in \mathcal{W}$  associated to  $f \in \mathcal{F}$  is naturally decomposed into the union of a *base*, *beams* and *plates*. The base is the closure of the planar surface consisting of  $f$  with the attaching zones of the handles removed. The beams are the lateral surfaces of the 1-handles minus the open attaching zones of the 2-handles and the plates are the lateral surfaces of the 2-handles, where each plate consists of two parallel discs. The beams are standardly embedded. An *extended beam* is the full lateral surface of a 1-handle. See Figure 6.

**Proposition 8.2.** *Let  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  be a  $F|W$  system on  $V_k$  with  $\mathcal{W}$  homotopic to  $\mathcal{F}$ . Then there exists a  $F|W$  system  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}')$  with  $\mathcal{W}'$  standardly concordant to  $\mathcal{F}$  and  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  S-equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}')$ .*

*Proof.* Let  $\mathcal{N}_f$  (resp.  $\mathcal{N}_w$ ) be a system of dual spheres to  $\mathcal{F}$  (resp.  $\mathcal{W}$ ), which exists by Lemma 4.22. If  $|\mathcal{F}| = n$ , then let  $f_t : \cup_{i=1}^n D^2 \rightarrow V_k$  be a regular homotopy [Sm] supported away from  $N(\mathcal{G} \cup \mathcal{F})$  such that  $f_0 = \mathcal{F}$  and  $f_1 = \mathcal{W}$ , where we abuse notation by identifying a map with its image. We will assume that all the finger moves occur at  $t = 1/4$  and all the Whitney moves when  $t = 3/4$ . Since the support of the finger moves can be chosen to lie in a small neighborhood of the union of  $\mathcal{F}$  and arcs we can assume that  $f_{.251}$  is *dual* to  $\mathcal{N}_f$ . This means that  $f_{.251}(D_i)$  intersects  $\mathcal{N}_f$  exactly once and at the component of  $\mathcal{N}_f$  that intersects  $f_0(D_i)$ . Since  $f_1$  is obtained by first applying Whitney moves to  $f_{.749}$  and then isotopy, it follows that  $f_{.749}$  is obtained from  $f_1$  by finger moves and isotopy. Thus we can assume that  $f_{.749}$  is dual to  $\mathcal{N}_w$  and that there is an ambient isotopy from  $f_{.251}$  to  $f_{.749}$  which starts out dual to  $\mathcal{N}_f$  and ends dual to  $\mathcal{N}_w$ .

Each finger move  $f_k$  creates two points of self intersection of  $f_{.251}$  which are then eliminated by replacing two discs on one sheet of  $f_{.251}$  by a tube  $T_k$  as in Figure 6 [MM] that lies very close to an arc  $\delta_k \subset f_{.251}$ . Let  $R_1$  denote the embedded surface obtained by modifying

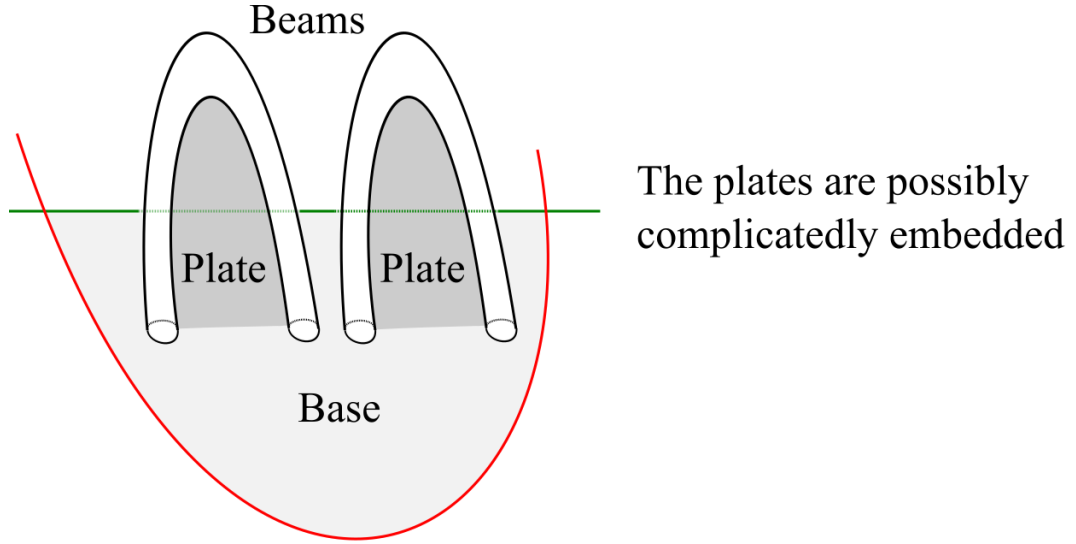


FIGURE 6. A standardly concordant Whitney disc

$f_{.251}$  by these tubings. The ambient isotopy induces a diffeomorphism  $g_1 : V_k \rightarrow V_k$  with  $g_1(f_{.251}) = f_{.749}$  and an isotopy of  $R_1$  to  $g_1(R_1) := R_2$ , where we can assume that  $g_1(T_k)$  is a tube lying close to  $g_1(\alpha_k)$ , i.e. we can assume that  $g_1$  takes a small regular neighborhood of  $\alpha_k$  containing  $T_k$  to a small regular neighborhood of  $g_1(\alpha_k)$ . The core circle of  $T_k$  bounds a small 2-disc  $A_k$  that intersects  $f_{.251}$  in a point  $a_k \in \alpha_k$ . Let  $g_1(A_k) := E_k$ . Since  $f_{.251}$  (resp.  $f_{.749}$ ) is dual to  $\mathcal{N}_f$  (resp.  $\mathcal{N}_w$ ) we obtain an embedded disc  $B_k$  (resp.  $F_k$ ) by tubing  $A_k$  (resp.  $E_k$ ) with a copy of a component of  $\mathcal{N}_f$  (resp.  $\mathcal{N}_w$ ). The tube follows an embedded arc in  $f_{.251}$  (resp.  $f_{.749}$ ) from  $a_k$  (resp.  $g_1(a_k)$ ) to a point near  $\mathcal{N}_f$  (resp.  $\mathcal{N}_w$ ). Think of  $F_k$  as a reimbedding of  $g_1(B_k)$ .

Thicken  $F_k$  to a 3-dimensional 2-handle  $\omega_k$  and let  $\mathcal{W}'$  be the result of embedded surgery of  $R_2$  along the  $\omega_k$ 's. By construction,  $\mathcal{W}'$   $\partial$ -coincides with  $\mathcal{W}$  and is dual to  $\mathcal{N}_w$ . By Lemma 4.24  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is S-equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W}')$ .

To complete the proof we show that up to isotopy  $\mathcal{W}'$  is standardly concordant to  $\mathcal{F}$ . As in §4.2 [MM] up to isotopy  $R_1$  is obtained from  $f_{.249}$  by embedded surgery along 3-dimensional 1-handles, one 1-handle for each finger move. See Figure 6 [MM]. Up to isotopy the thickened  $B_k$ 's denoted  $\tau_k$ 's are cancelling 2-handles. Now reimbed each  $\tau_k$  using  $g_1^{-1}(\omega_k)$ . Since  $\mathcal{W}'$  is isotopic to the surface obtained by embedded surgery to  $R_1$  along  $g_1^{-1}(\omega_k)$  the result follows.  $\square$

## 9. FINGER|WHITNEY-CARVING/SURGERY PRESENTATIONS OF SCHOENFLIES SPHERES

We start by defining the notion of a carving/surgery presentation of a Schoenflies sphere. We then define a specialized form of this presentation called a  $F|W$ -carving/surgery presentation. The main result of the next section is that every Schoenflies sphere has a  $F|W$ -carving/surgery presentation.

**Definition 9.1.** Let  $\mathbb{S}_0 \subset S^4$  denote the standard 3-sphere and let  $X_S^0, X_N^0$  denote its 4-ball closed complementary regions. A *carving/surgery presentation* of a Schoenflies sphere  $\mathbb{S}$  in the 4-sphere with closed complementary regions  $\Delta_S$  and  $\Delta_N$  consists of



i) a framed link  $L = \{k_1, \dots, k_n\} \subset \mathbb{S}_0$  that surgers  $\mathbb{S}_0$  to  $S^3$  such that each component  $k_i$  bounds an embedded disc  $D_{k_i}$  such that  $D_{k_i} \cap \mathbb{S}_0 \subset N^3(L)$  and induces the given framing on  $k_i$ . Here  $N^3(L)$  denotes a regular neighborhood of  $L \subset \mathbb{S}_0$ . If a neighborhood of  $\partial D_{k_i} \subset X_S^0$  (resp.  $X_N^0$ ), then label  $k_i$  with a dot (resp. 0).

ii) If  $\text{int}(D_{k_i}) \cap N^3(k_j) \neq \emptyset$ , then the components of  $D_{k_i} \cap N^3(k_j)$  bound pairwise disjoint discs in  $D_{k_i}$  that are parallel to  $D_{k_j}$  and contained in a small neighborhood of  $D_{k_j}$ . This induces a partial order on the  $k_i$ 's and the  $D_{k_i}$ 's, with the minimal elements corresponding to discs  $D_k$  such that  $D_k \cap \mathbb{S}_0 = k$ . Call a  $k \in L$  or its corresponding disc  $D_k$  a *depth- $m$  element* if there is a maximal length sequence  $k = k_{i_0} > k_{i_1} > k_{i_2} > \dots > k_{i_{m-1}}$ . Let  $M$  be the maximal depth over all the components of  $L$ .

iii) Let  $X_S^1 \subset S^4$  denote the compact manifold obtained from  $X_S^0$  by carvings [Ak] corresponding to the minimal discs contained in  $X_S^0$  and by adding 2-handles corresponding to the minimal discs contained in  $X_N^0$ . Let  $X_N^1$  denote the closed complementary region of  $X_S^1$ . Let  $X_S^2 \subset S^4$  denote the compact manifold obtained from  $X_S^1$  by carving the depth-2 discs lying in  $X_S^1$  and adding 2-handles corresponding to the depth-2 discs lying in  $X_N^1$ . Let  $X_N^2$  denote the closed complementary region to  $X_S^2$ . In a similar manner construct  $X_S^3, \dots, X_S^M$  and  $X_N^3, \dots, X_N^M$ .

iv) Define  $\Delta_S = X_S^M$ ,  $\Delta_N = X_N^M$  and  $\mathbb{S} = \partial\Delta_S = \partial\Delta_N$ . By construction  $\mathbb{S} = S^3$  and  $\Delta_S$  and  $\Delta_N$  are its Schoenflies balls.

**Definition 9.2.** If  $k$  is depth  $m$  and labeled with a 0, then let  $D_k^0$  denote the associated 2-handle attached to  $X_S^{m-1}$ , while if labeled with a dot, then let  $D_k^\bullet$  denote the associated 2-handle attached to  $X_N^{m-1}$ . We will abuse notation by calling  $D_k^0$  (resp.  $D_k^\bullet$ ) a *2-handle of  $\Delta_S$  (resp.  $\Delta_N$ )* and a *carving of  $\Delta_N$  (resp.  $\Delta_S$ )*. Viewed in  $\Delta_S$ ,  $D_k^0$  will be a 2-handle with a  $D^2 \times U$  deleted, where  $U$  is a possibly empty open subset of the cocore. E.g. if  $M - m = 1$ , then  $D_k^0$  may have finitely many parallel copies of its core carved out.

**Notation 9.3.** Partial orders will often be depicted as forests with oriented edges. Recall from Notation 3.7 that  $V_k = S^1 \times S^3 \#_k S^2 \times S^2$  with  $V_\infty$  its universal cover where the lifts of the  $S^2 \times S^2$  summands are denoted  $\widetilde{S^2 \times S^2}_i, i \in \mathbb{Z}$ . In what follows  $\widetilde{\mathbb{S}}_0 \subset V_\infty$  will denote a 3-sphere of the form  $pt \times S^3$  that separates the  $\widetilde{S^2 \times S^2}_i$ 's where  $i > 0$  from the  $\widetilde{S^2 \times S^2}_j$ 's where  $j \leq 0$ . To make the notation more symmetric about 0, reindex  $\widetilde{S^2 \times S^2}_i$  as  $\widetilde{S^2 \times S^2}_{i-1}$  when  $i \leq 0$ . Similarly reindex  $\widetilde{R}_i, \widetilde{G}_i$ , etc when  $i \leq 0$ . A finger (resp. Whitney) disc from  $\widetilde{R}_i$  to  $\widetilde{G}_j^{\text{std}}$  will be denoted  $\widetilde{f}_{ij}$  (resp.  $\widetilde{w}_{ij}$ ), though there may be more than one such disc for a given  $i$  and  $j$ .

**Definition 9.4.** We say that  $D_i$  *nestjs in*  $D_j$  if some component of  $D_i \cap N(k_j)$  is outermost in  $D_i \cap (\cup_{p \neq i} N(k_p))$ . We also say that  $k_i$  or  $D_i$ 's 2-handle or carving *nestjs in*  $k_j$  or  $D_j$ 's carving or 2-handle. In this case we also say that  $D_j$  or its 2-handle or carving or  $k_j$  *lodges*  $D_i$  or its carving or 2-handle and  $k_j$ .

**Definition 9.5.** Given the pairwise disjoint knots  $k_1, \dots, k_r$  in  $\mathbb{S}_0$ , then  $k'_1, \dots, k'_r$  are called *linking circles* if they bound pairwise disjoint discs in  $\mathbb{S}_0$  that  $\delta_{ij}$  intersect the  $k_i$ 's.

**Definition 9.6.** A  $F|W$ -carving/surgery presentation is a carving/surgery presentation whose link  $L$  is a disjoint union of *knots*  $L_k$  and their linking circles such that all the linking circles are 0-framed. Furthermore  $L_k = B_L \sqcup S_L \sqcup N_L$  with the partial order and 0, dot labeling arising as follows.

1a)  $B_L$  is the unlink  $U = \{b_{i_1}, \dots, b_{i_p}\}$  together with its linking circles. Here  $i_j \in \mathbb{Z}_{\neq 0}$  and  $i_j \neq i_k$  unless  $j = k$ . If  $i < 0$  (resp.  $> 0$ ), then  $b_i$  is labeled with a 0 (resp. dot) and its linking circle  $b'_i$  is labeled with a dot (resp. 0). Within  $B_L$ , there is a directed edge from each dotted linking circle to each knot with label 0 and a directed edge from each linking circle with label 0 to each dotted knot.

*Motivation* (where this comes from): A  $b_i, i < 0$  (resp.  $i > 0$ ) will arise if there is a finger disc of the form  $\tilde{f}_{ji}$  with  $j > 0$  (resp.  $j < 0$ ).

1b)  $S_L = \sqcup S_{ij}$  where  $i < 0$  and  $j \in \mathbb{Z}_{\neq 0}$ . Associated to each  $S_{ij}$  is a finite union  $\mathcal{W}_{ij}$  of 2-discs each of which contains finitely many pairwise disjoint simple closed curves whose union is denoted  $S_{ij}^k$ . The elements of  $S_{ij}^k$  are in 1-1 correspondence with knots in  $\mathbb{S}_0$  and each knot comes with a linking circle.  $S_{ij}$  is the union of these knots and their linking circles.

*Motivation* (where this comes from): The discs associated to  $S_{ij}$  are the  $\tilde{w}_{ij}$  discs with  $i < 0$  and the simple closed curves are intersections of the  $\tilde{w}_{ij}$ 's with  $\tilde{\mathbb{S}}_0$ .

1b) continued: The partial ordering restricted to the knots is induced by the inclusion relation of the simple closed curves within the discs, with the innermost curves being the minimal ones. The linking circles are given the opposite partial ordering. If two knots are connected by an edge, then one is labeled with a 0 and the other with a dot, subject to the condition that the maximal knots are labeled with 0's. If a knot is labeled with a 0 (resp. dot), then its linking circle is labeled with a dot (resp. 0). See Figure 7. The boxes on the knots remind us that they may be knotted and linked with other knots of  $L$ .

1c)  $N_L = \sqcup N_{ij}$  where  $i > 0$  and  $j \in \mathbb{Z}_{\neq 0}$ . Associated to each  $N_{ij}$  is a finite union  $\mathcal{W}_{ij}$  of discs each of which contains pairwise disjoint simple closed curves whose union is denoted  $N_{ij}^k$ . The elements of  $N_{ij}^k$  are in 1-1 correspondence with a set of knots in  $\mathbb{S}_0$  and each knot comes with a linking circle.  $N_{ij}$  is the union of the knots and their linking circles.

*Motivation* (where this comes from): The discs associated to  $N_{ij}$  are the  $\tilde{w}_{ij}$  discs with  $i > 0$  and the simple closed curves are intersections of the  $\tilde{w}_{ij}$ 's with  $\tilde{\mathbb{S}}_0$ .

1c) continued: The partial ordering restricted to the knots is induced by the inclusion relation of the simple closed curves within the discs, with the innermost curves being the minimal ones. The linking circles are given the opposite partial ordering. If two knots are connected by an edge, then one is labeled with a 0 and the other with a dot, subject to the condition that the maximal knots are labeled with dots's. If a knot is labeled with a 0 (resp. dot), then its linking circle is labeled with a dot (resp. 0). See Figure 7.

1d) Let  $\alpha$  be the knot of  $S_{ij}$  or  $N_{ij}$  associated to the element  $\beta \in N_{ij}^k$  or  $S_{ij}^k$ . We say that both  $\alpha$  and  $\beta$  are at *level*  $n$  if there exists an arc from  $\beta$  to  $\partial\mathcal{W}_{ij}$  which intersects  $N_{ij}^k$   $n$  times where  $n$  is the minimal possible.

2a) *Order relations involving  $B_L$  and  $N_L \cup S_L$* : For every  $i, j, k \in \mathbb{Z}_{\neq 0}$  with  $j < 0$  and  $k > 0$  construct directed edges according to Figure 8 a). I.e. for each maximal knot of  $N_{kj}$  there is a directed edge from its linking circle to the linking circle  $b'_j$  of  $b_j$ . For every maximal knot of  $S_{ji}$  there is a directed edge from  $b'_j$  to the knot.

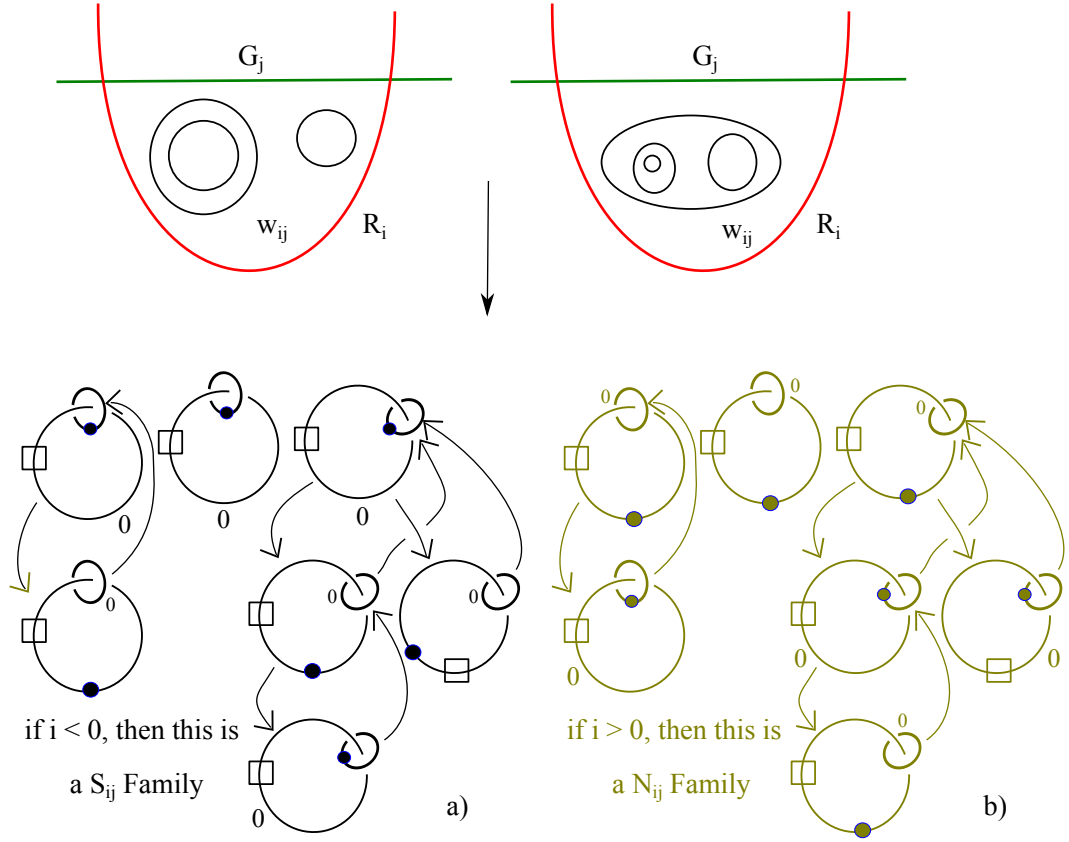


FIGURE 7. Constructing the  $S_{ij}$  and  $N_{ij}$  Families from Intersection Data

In addition, for every knot of  $N_L \cup S_L$  labeled with a dot, maximal or not, construct a directed edge to each  $b_j$  where  $j < 0$ .

For every  $i, j, k \in \mathbb{Z}_{\neq 0}$  with  $j > 0$  and  $i < 0$  construct directed edges according to Figure 8 b). I.e. For each maximal knot of  $S_{ij}$  there is a directed edge from its linking circle to  $b'_j$ . For each maximal knot of  $N_{jk}$  there is a directed edge from  $b'_j$  to the knot.

In addition, for every knot of  $N_L \cup S_L$  labeled with a 0, maximal or not, construct a directed edge to each  $b_j$  where  $j > 0$ .

2b) *Additional order relations involving  $S_L$* : For every maximal knot  $k$  of  $S_{ji}$  and every maximal knot  $k_1$  of  $S_{ik}$  construct a directed edge from the linking circle  $k'$  of  $k$  to the knot  $k_1$ . In particular for every maximal knot  $k$  of  $S_{ii}$  with linking circle  $k'$  there is a directed edge from  $k'$  to  $k$  and if  $k_1$  is another maximal knot of  $S_{ii}$  with linking circle  $k'_1$ , there is a directed edge from  $k'_1$  to  $k$  and a directed edge from  $k'$  to  $k_1$ .

2c) *Additional order relations involving  $N_L$* : For every maximal knot  $k$  of  $N_{ji}$  and every maximal knot  $k_1$  of  $N_{ik}$  construct a directed edge from the linking circle  $k'$  of  $k$  to the knot  $k_1$ . In particular for every maximal knot  $k$  of  $N_{ii}$  there is a directed edge from  $k'$  to  $k$  and if  $k_1$  is another maximal knot of  $N_{ii}$  with linking circle  $k'_1$ , there is a directed edge from  $k'_1$  to  $k$  and a directed edge from  $k'$  to  $k_1$ .

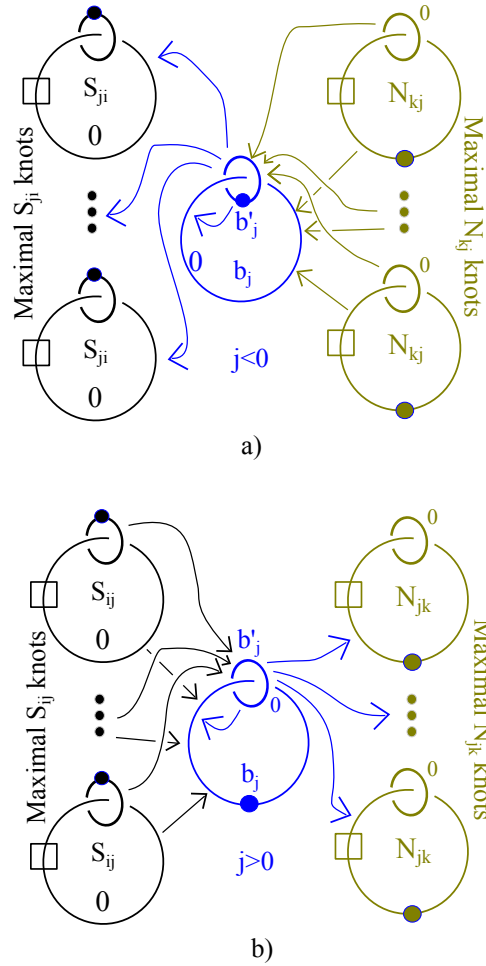


FIGURE 8. Order relations involving  $B_L$ . Non maximal knots not shown.

This completes the description of the additional combinatorial structure needed for a carving/surgery presentation to be a  $F|W$ -carving/surgery presentation, subject to checking that the order relation is a partial order.

**Lemma 9.7.** *The order relation on the link  $L$  of a  $F|W$ -carving/surgery presentation is a partial order.*

*Proof.* The knots of  $B_L$  are minimal elements. After deleting these elements, all remaining directed edges from the knots of  $S_L$  (resp.  $N_L$ ) go to knots of  $S_L$  (resp.  $N_L$ ) and the ordering of these knots is induced from the partial ordering on embedded simple closed curves on discs. Thus it suffices to prove the lemma for the ordering restricted to the linking circles of  $L$ . With respect to that ordering the linking circles of  $B_L$  are minimal elements and again the ordering of what remains is induced from the partial ordering on embedded simple closed curves on discs.  $\square$

**Definition 9.8.** We continue to call a carving/surgery presentation a  $F|W$ -carving/surgery presentation (FWCS) if it satisfies the conditions of Definition 9.6 except that the directed edges are a proper subset of those stated in that definition.

**Remark 9.9.** In the proof of Theorem 10.1 proper subsets may occur for the directed edges from  $g'_j$ 's to  $g_\ell$ 's and those of the second and fourth paragraphs of 2a).

**Definition 9.10.** An *optimized* FWCS-presentation is a FWCS-presentation with the following additional features.

- i) The knots of  $S_{ij}$  and  $N_{ij}$  are of level at most 2.
- ii) If  $k \in L_k$ , then  $D_k$  induces the 0-framing on  $k$ .
- iii) The knots  $S_L \cup N_L$  are the disjoint union of  $\mathcal{A}$  and  $\mathcal{B}$  where  $\mathcal{A} \cup B_L$  and  $\mathcal{B} \cup B_L$  are unlinks in  $S_0$ .
- iv)  $\mathcal{B}$  includes all the level-2 knots and if  $k \in \mathcal{B}$ , then  $k$  is a minimal element with respect to the partial order on  $L$ .

## 10. SCHOENFLIES SPHERES HAVE FINGER|WHITNEY-CARVING/SURGERY PRESENTATIONS

The following is the main result of this section.

**Theorem 10.1.** *Every smooth 3-sphere in the 4-sphere has a FWCS-presentation.*

*Proof.* By Proposition 1.10 every 3-sphere  $\Sigma' \subset S^4$  corresponds to the S-equivalence class of some  $\phi \in \text{Diff}_0(S^1 \times S^3)$ . Viewing  $S^4$  as  $(\mathbb{R} \times S^3) \cup \{S, N\}$ , then  $\Sigma'$  is isotopic to  $\tilde{\phi}(\text{pt} \times S^3)$  where  $\tilde{\phi}$  is the lift to  $S^1 \times S^3$ . By Lashoff - Shaneson [LS] and Sato [Sa] there is a pseudo-isotopy  $f$  from id to  $\phi$  which by [HW] arises from a 1-parameter Hatcher-Wagoner family  $(q_t, v_t)$ . We will assume that  $(q_t, v_t)$  induces a  $F|W$  system satisfying the conclusion of Proposition 8.2 and that  $(q_t, v_t)$  has been normalized as in the second and third paragraphs of the proof of Theorem 2.5. In what follows  $V$  will denote  $S^1 \times S^3$  and  $V_k$  will denote  $S^1 \times S^3 \#_k S^2 \times S^2$  where  $k$  is the number of components of the nested eye. It suffices to prove the theorem for the 3-sphere  $\Sigma$  obtained from  $\Sigma'$  by reversing orientation, which by Proposition 1.10 is the class of  $\tilde{\phi}^{-1}(\text{pt} \times S^3)$ .

**Step 1:** Show how to construct  $\phi^{-1}(U)$  from the  $F|W$ -system where  $U$  is a closed submanifold of  $V$ .

We view the 1-parameter family  $(q_t, v_t)$  as a smoothly varying family of handle structures  $h_t$  on  $V \times I$  where  $h_{1/4}$  corresponds to  $k$  standardly cancelling 2 and 3-handles and for  $t \in [1/4, 3/4]$  the handle structure changes according to the path of 2-sphere boundaries of the cores of the attaching 3-handles.

We introduce some terminology to keep track of the data, in particular both before and after the 2-handle attachments. First  $h_t$  denotes the handle structure on  $V \times I \times t$ . For  $t \in [1/4, 3/4]$  the 2-handles are attached to  $V \times [0, 1/4] \times t$  along a set of  $k$  0-framed simple closed curves  $\Omega \times 1/4 \times t$ , where  $\Omega = \{\omega_1, \dots, \omega_k\}$  and the  $\omega_i$ 's bound pairwise disjoint discs  $\mathcal{D} = \{D_1, \dots, D_k\}$ . We abuse notation by also viewing  $\Omega$  and the  $D_i$ 's as subsets of  $V \times 0 \times t$ . Let  $V_k \times t$  denote the result of attaching these 2-handles. This  $V_k \times t$  corresponds to the  $V_k \times 1/4 \times t$  in the proof of Theorem 2.5. With terminology as in §2,  $\mathcal{G}^{\text{std}} = \{G_1^{\text{std}}, \dots, G_k^{\text{std}}\} \subset V_k \times t$  denotes the boundaries of the 2-handle cocores, where  $G_i^{\text{std}}$  is the standard green sphere in the  $i$ 'th  $S^2 \times S^2$  factor. It is of the form  $S^2 \times \text{pt}$ . When  $t = 1/4$ , the 3-handles are attached along the set of spheres  $\mathcal{R}^{\text{std}} = \{R_1^{\text{std}}, \dots, R_k^{\text{std}}\}$  where  $R_i^{\text{std}}$  is the  $i$ 'th standard red sphere of the form  $\text{pt} \times S^2$  in the  $i$ 'th  $S^2 \times S^2$  factor and  $R_i^{\text{std}}$

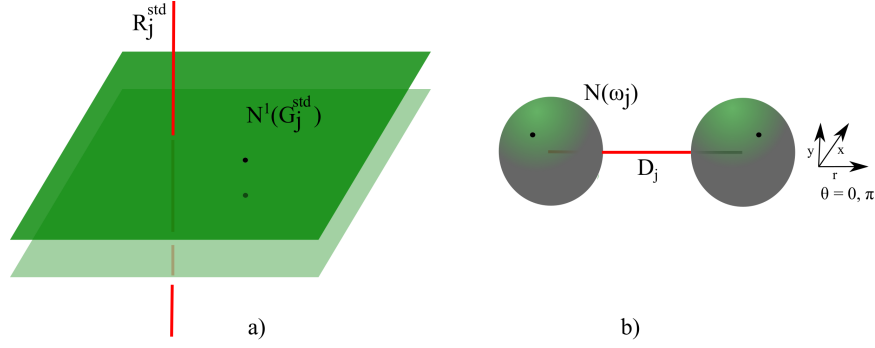


FIGURE 9. Sections and Fibers

flows to  $D_i \subset V \times 0 \times 1/4$  under  $-v_{1/4}$ . Fix regular neighborhoods  $N(\Omega)$  and  $N(\mathcal{G}^{\text{std}})$  so that  $V_k \times t \setminus \text{int}(N(\mathcal{G}^{\text{std}}))$  is diffeomorphic to  $V \times 0 \times t \setminus \text{int}(N(\Omega))$  with the diffeomorphism  $\lambda$  induced by  $-v_t$ . Note that both are diffeomorphic to  $S^1 \times S^3 \#_k S^2 \times D^2$ . Thanks to our initial normalization, much of the data corresponding to the projections to  $V_k$  and  $V$  are independent of  $t \in [1/4, 3/4]$ , e.g.  $\Omega, \mathcal{G}^{\text{std}}, \mathcal{R}^{\text{std}}$ , the  $D_i$ 's and  $\lambda$ . To help keep track of  $\lambda|_{N(\mathcal{G}^{\text{std}})}$  we recall the

*Section-Fiber Rule:* For the 2-handle  $D^2 \times D^3$  with  $\partial \text{ core} = \omega_i = \partial D^2 \times 0$  and  $\partial \text{ cocore} = G_i^{\text{std}} = 0 \times \partial D^3$ , with  $\omega_i$  viewed in  $V \times 0 \times t$  and  $G_i^{\text{std}}$  viewed in  $V_k \times t$ , then  $\partial D^2 \times \partial D^3$  can be simultaneously viewed as  $\partial N(G_i^{\text{std}}) = N^1(G_i^{\text{std}})$  and as  $\partial N(\omega_i) = N^1(\omega_i)$  with  $\lambda$  equating the two. Here  $N^1$  denotes the unit normal bundle of the space in question. Under  $\lambda$  a section  $\text{pt} \times G_i^{\text{std}}$  of  $N^1(G_i^{\text{std}})$  is identified to a normal fiber  $\text{pt} \times S^2$  of  $N^1(\omega_i)$  and a normal fiber  $S^1 \times \text{pt}$  of  $N^1(G_i^{\text{std}})$  is identified with a section  $\omega_i \times \text{pt}$  of  $N^1(\omega_i)$ .

Figure 9 a) shows a 3-dimensional slice of  $N^1(G_j^{\text{std}}) \cup R_j^{\text{std}}$ . The two points in a) are the intersection with a circle fiber and the dark 2-sphere on top and the light one below are 2-sphere sections of  $N^1(G_j^{\text{std}})$ . Construct  $N(\omega_j)$  by taking a product of a 3-dimensional regular neighborhood of  $\omega_j$  with  $[-1, 1]$ . Here the sections and fibers of  $N^1(\omega_j)$  are evident. Alternatively we can construct  $N(\omega_j)$  using cylindrical coordinates  $(x, y, r, \theta)$  and Figure 9 b) shows the  $\theta = 0, \pi$  slices with two 2-sphere fibers and the intersection with a section.

Let  $\mathcal{R}_t := \{R_{1,t}, \dots, R_{k,t}\} \subset V_k \times t$  denote the boundary of the cores of the 3-handles, where  $\mathcal{R}_{1/4} = \mathcal{R}^{\text{std}}$ . When projected to  $V_k$ ,  $\mathcal{R}_t$  first undergoes finger moves with  $\mathcal{G}^{\text{std}}$  and then Whitney moves according to its  $F|W$  system. Let  $t_0 \in [1/4, 3/4]$  be just after the Whitney moves are completed. Let  $\mathcal{R}_t^0 = \{R_{1,t}, \dots, R_{k,t}\}$  denote  $\lambda(\mathcal{R}_t \setminus \text{int } N(\mathcal{G}^{\text{std}}))$ . We can assume that  $\mathcal{R}_t \cap N(\mathcal{G}^{\text{std}})$  is a union of fibers of  $N(\mathcal{G}^{\text{std}})$  except during the finger and Whitney moves. Figure 10.2 shows  $\mathcal{R}_{t \pm \epsilon}$  and  $\mathcal{R}_{t \pm \epsilon}^0$  just before and after a finger move. We can assume that for  $t \in [1/4, t_0]$ ,  $\mathcal{R}_t^0$  coincides with  $\mathcal{R}_{1/4}^0$  near  $\partial \mathcal{R}_{1/4}^0$ , where  $\mathcal{R}_{1/4}^0 = \mathcal{D} \setminus \text{int } N(\Omega)$ . Having completed the Whitney moves,  $\mathcal{R}_{t_0}^0$  is a union of discs. In what follows we abuse notation by viewing  $\mathcal{R}_t \subset V_k$  and  $\mathcal{R}_t^0 \subset V$ .

Under isotopy extension the path  $\mathcal{R}_t$  extends to an ambient isotopy  $\eta_t : V_k \rightarrow V_k, t \in [1/4, t_0]$ . Let  $U := U_{1/4}^0 \subset V$ , be a closed cod- $\geq 1$  submanifold disjoint from  $N(\mathcal{D})$  and define  $U_{1/4} = \lambda^{-1}(U_{1/4}^0)$ . Note that  $U_{1/4}$  is disjoint from  $N(\mathcal{G}^{\text{std}} \cup \mathcal{R}^{\text{std}})$ . Define  $U_t = \eta_t(U_{1/4})$ . We

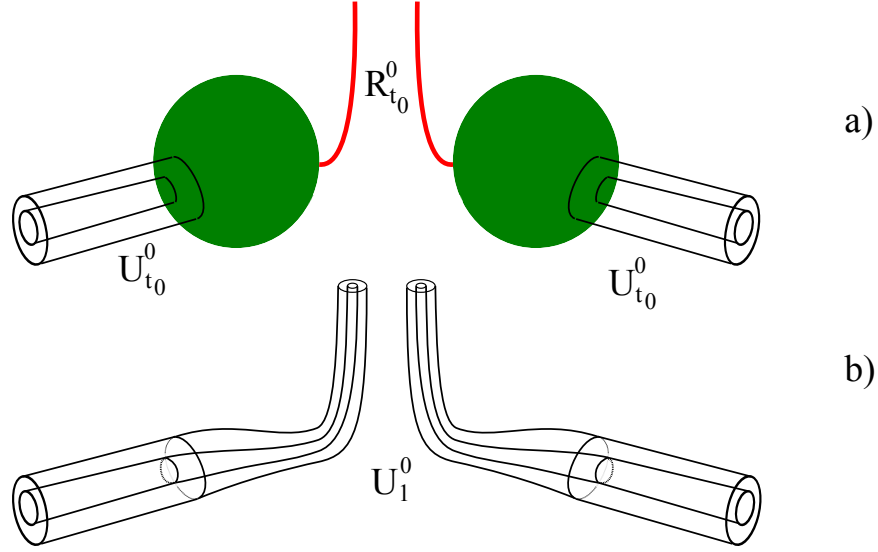


FIGURE 10. A Glissade

can assume that  $U_{t_0} \cap N(\mathcal{G}^{\text{std}})$  is a union of normal discs. By construction it is disjoint from  $\eta_t(N(\mathcal{R}^{\text{std}}))$ . Let  $U_t^0 := \lambda(U_t \setminus \text{int}(\mathcal{G}^{\text{std}}))$ .

We now cancel the 2 and 3-handles of  $h_{t_0}$  or equivalently introduce deaths to extend the 1-parameter family  $(q_t, v_t), t \in [0, t_0]$  to a Hatcher - Wagoner family  $(q'_t, v'_t), t \in [0, 1]$ , where  $(q'_t, v'_t) = (q_t, v_t)$  for  $t \in [0, t_0]$  and  $(q'_t, v'_t)$  has no excess 3/2 intersections in  $[t_0, 1]$  and  $(q_1, v_1)$  is nonsingular. The induced pseudo-isotopy is from  $\text{id}$  to a  $\phi' : V \rightarrow V$  which by Lemma 2.6 is isotopic to  $\phi$ . By Lemma 2.15 the isotopy class of  $\phi'$  is independent of the isotopy extension  $\eta_t$ .

Under cancellation of the 2 and 3-handles of  $h_{t_0}$  as in [Mi],  $U_{t_0}^0$  is modified as follows to obtain  $U_1^0 = \phi^{-1}(U)$ , up to isotopy. If  $C$  is a normal fiber of  $U_{t_0} \cap N^1(\mathcal{G}^{\text{std}})$ , then  $\lambda(C)$  is a section of  $N^1(\Omega)$  that is capped off by the union of an annulus in  $N(\Omega)$  and a parallel copy of a component of  $\mathcal{R}_{t_0}^0$ . Note that if  $K$  is any compact subset of fibers of  $N^1(\mathcal{G}^{\text{std}})$  disjoint from  $\mathcal{R}^{\text{std}}$ , e.g.  $\lambda^{-1}(\partial U_{t_0}^0)$  then  $\lambda(K)$  can be capped off by a corresponding compact set of annuli and subsets of components of  $\mathcal{R}_{t_0}^0$ . See Figure 10. This completes the construction of Step 1.

**Definition 10.2.** We call the operation of replacing  $U_{t_0}^0$  by  $U_1^0$  a *glissade*.

**Notation 10.3.** As in Notation 9.3 we modify Notation 3.7 by reindexing  $\widetilde{S^2} \times \widetilde{S^2}_i, \widetilde{R^{\text{std}}}_i, \widetilde{G^{\text{std}}}_i$ , etc. so that each  $i \leq 0$  is replaced by  $i-1$ . Therefore  $\tilde{\pi}^{-1}[1-2\epsilon, -1+2\epsilon] = [1-2\epsilon, -1+2\epsilon] \times S^3$  with  $\rho$  modified accordingly. In particular, when compared with the original  $\rho$ , our new  $\rho, \rho^{-1}$  when restricted to  $[1-2\epsilon, -1+2\epsilon] \times S^3$  are contractions in the first factor.

Let  $\tilde{h}_t$  denote the lift of the handle structure  $h_t$  to  $\tilde{V} \times I \times t$ . In a similar manner we reindex its data. Let  $\mathbb{S}_0 := 0 \times S^3 \subset \tilde{V}$  and  $\mathbb{S}'_0 := 0 \times S^3 \subset \tilde{V}_k$ . With  $S^4$  identified with  $\tilde{V} \cup \{S, N\}$ ,  $\mathbb{S}_0$  is our standard 3-sphere in  $S^4$ .

**Step 2:** Apply the construction of Step 1 to  $\mathbb{S}_0$  to construct a sphere  $\mathbb{S}_1$  isotopic to  $\phi^{-1}(\mathbb{S}_0)$ .

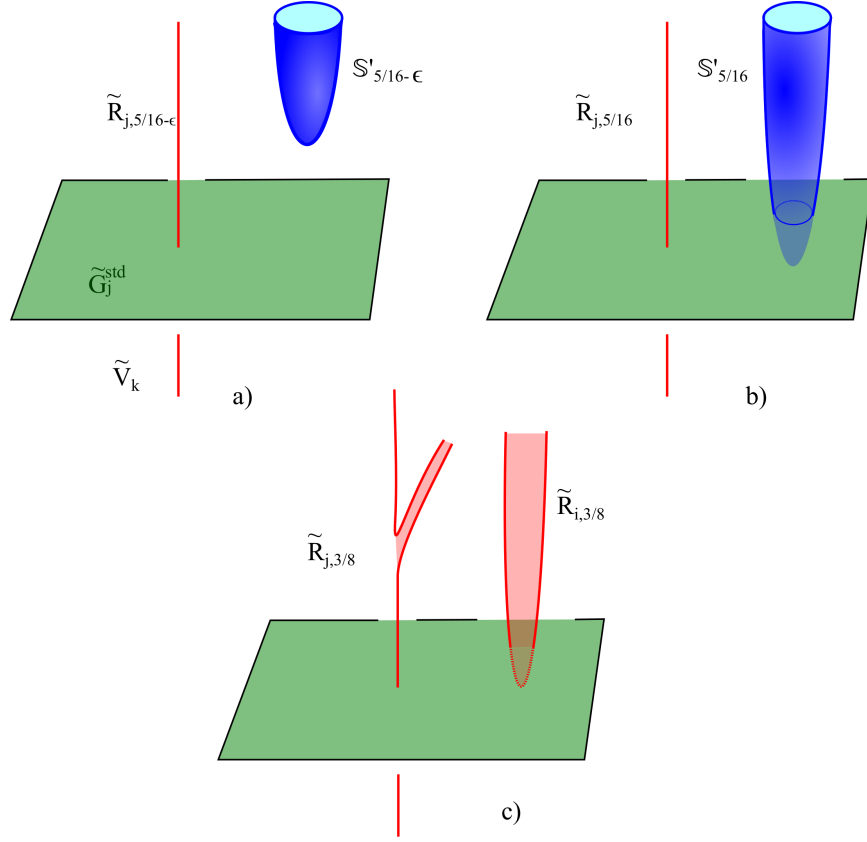


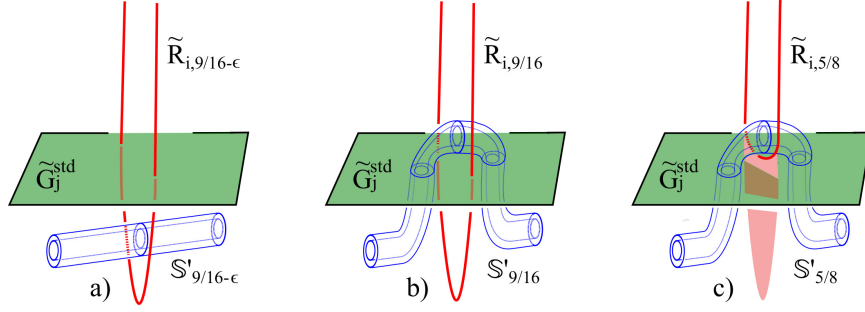
FIGURE 11. Preparing for and doing the Finger move, as seen in  $\tilde{V}_k$

Let  $\mathbb{S}'_{1/4} = \mathbb{S}'_0 \subset \tilde{V}_k$  and  $\mathbb{S}_{1/4} = \mathbb{S}_0 \subset \tilde{V}$ . In what follows we will assume that the finger (resp. Whitney) moves occur during  $t \in (3/8 - \epsilon, 3/8)$  (resp.  $t \in (5/8 - \epsilon, 5/8)$ ) and  $t_0 = 5/8$  and that the finger moves associated to a given arm are done together to have image that arm, hand and fingers. We will also assume that the handle cancellations occur during  $t \in (11/16 - \epsilon, 11/16)$  and so  $\mathbb{S}_{11/16} = \mathbb{S}_1$ . We now construct an explicit isotopy  $\mathbb{S}'_t \subset \tilde{V}_k$  with  $\tilde{\mathcal{R}}_t \cap \mathbb{S}'_t = \emptyset$  for  $t \in [1/4, t_0]$  and use it to construct  $\mathbb{S}_1$ .

i) *Prepare for the finger moves:* For each arm-hand of  $\tilde{\mathcal{F}}$  from  $\widetilde{R}^{\text{std}}_i$  to  $\widetilde{G}^{\text{std}}_j$  with  $i < 0$  and  $j > 0$  isotope  $\mathbb{S}'_{1/4}$  to  $\mathbb{S}'_{5/16}$  by doing a finger move from  $\mathbb{S}'_{1/4}$  to  $\widetilde{G}^{\text{std}}_j$  so that  $\mathbb{S}'_{5/16}$  is disjoint from the track of that arm-hand. This isotopy creates a torus component of  $\mathbb{S}'_{5/16} \cap \partial N(\widetilde{G}^{\text{std}}_j)$ . The corresponding passage from  $\mathbb{S}_{1/4}$  to  $\lambda(\mathbb{S}'_{5/16} \setminus \text{int } N(\widetilde{G}^{\text{std}})) := \mathbb{S}_{5/16}$  involves the removal of an open solid torus. In a similar manner isotope  $\mathbb{S}'_{1/4}$  to  $\mathbb{S}'_{5/16}$  when  $i > 0$  and  $j < 0$ . The supports of all these isotopies need to be disjoint. Figures 11 a), b) show the before and after local pictures in  $\tilde{V}_k$ , i.e. at times  $t = 5/16 - \epsilon$  and  $t = 5/16$  where  $5/16 - \epsilon$  is a time just before  $\mathbb{S}'_t$  locally intersects  $\widetilde{G}^{\text{std}}$ .

ii) *Do the finger moves:* For  $t \in [3/8 - \epsilon, 3/8)$  isotope  $\tilde{\mathcal{R}}_t$  according to its arms, hands and fingers. See Figure 11 c). Here  $\mathbb{S}'_{3/8}$  and  $\mathbb{S}_{3/8}$  are not shown.



FIGURE 12. Preparing for and doing the Whitney move, as seen in  $\tilde{V}_k$ 

iii) *Prepare for the Whitney moves:* If  $\tilde{w}_{ij} \subset \tilde{V}_k$  is a Whitney disc between  $\tilde{R}_{i,1/2}$  and  $\widetilde{G_j^{\text{std}}}$ , then  $\tilde{w}_{ij} \cap \mathbb{S}'_{1/2}$  is eliminated by an isotopy  $\mathbb{S}'_t$ ,  $t \in [(9/16 - \epsilon), 9/16)$  as in Figures 12 a), b). Note that each component of  $\tilde{w}_{ij} \cap \mathbb{S}'_{1/2}$  gives rise to two torus components of  $N^1(\widetilde{G_j^{\text{std}}}) \cap \mathbb{S}'_{9/16}$ . Also that both  $\mathbb{S}'_t$  and  $\tilde{R}_{i,t}$  extend into the past and the future.

iv) *Do the Whitney moves:* For  $t \in (5/8 - \epsilon, 5/8)$  isotope  $\tilde{R}_t$  according to its Whitney discs to obtain  $\tilde{R}_{5/8}$ . See Figure 12 c). The shaded region indicates the local projection of  $\tilde{R}_{i,5/8}$  to the present.

v) *Do the glissade:* We can assume that  $\mathbb{S}'_{5/8} \cap N(\widetilde{G_j^{\text{std}}})$  is a union of normal discs. If  $C \subset N^1(\widetilde{G_j^{\text{std}}})$  is the boundary of one such normal disc, then under the glissade  $\lambda(C)$  is capped off by a disc which is the union of an annulus  $\subset N(\tilde{w}_j)$  and a copy of  $\tilde{R}_{j,5/8}^0$ . This completes the construction of Step 2.

Since each  $\tilde{R}_{5/8}$  is isotopic to  $\tilde{R}_{1/4}$ , Theorem 10.1 [Gal] applies to any finite subset. Applying  $\tilde{\lambda}$  to this isotopy we obtain the following result.

**Lemma 10.4.** *For any finite set  $J \subset \mathbb{Z}$ , there exists an ambient isotopy  $\kappa_t$  of  $\tilde{V}$  fixing  $\cup_{j \in J} N(\tilde{w}_j)$  pointwise such  $\kappa_0 = \text{id}$  and  $\kappa_1(\cup_{j \in J} \tilde{R}_{5/8}^0) = \cup_{j \in J} \tilde{R}_{1/4}^0$ .  $\square$*

**Remark 10.5.** Actually the construction of Step 2 can be done at the  $V$  and  $V_k$  level so that there is a  $\mathbb{Z}$ -equivariant ambient isotopy of  $\tilde{R}_{5/8}^0$  to  $\tilde{R}_{1/4}^0$  that fixes  $N(\tilde{\Omega})$  pointwise.

**Step 3:** Learn to work with anellini discs.

**Definition 10.6.** An *anellini disc* is a  $D^2 \times S^1$  viewed as a  $N^1(D^2)$ .

An anellini disc is the 3-dimensional analogue of an annulus viewed as a thin tube. As it is often beneficial to work with surfaces having subsurfaces a union of tubes, anellini discs are useful for working with 3-dimensional submanifolds of 4-manifolds.

Our next definition formalizes the well-known operation of Figure 13 in terms of carvings, surgeries and anellini discs.

**Definition 10.7.** Figure 13 a) schematically shows a disc  $D$  intersecting  $S^3 \subset S^4$  in the knot  $k$ , which bounds the subdisc  $D_1 \subset X_0^N$ , the northern 4-ball.  $S^3$  is isotopic to  $S_1^3$ , the

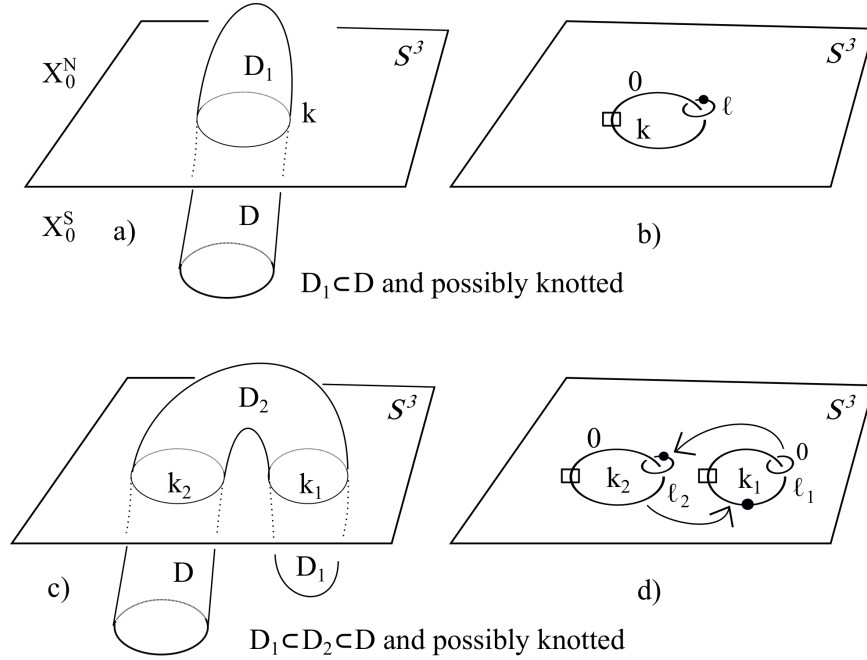


FIGURE 13. Anellini compression

result of embedded surgery along a 2-handle with core  $D_1$  and carving the standard disc  $D_\ell$  bounded by the linking circle  $\ell$ . See Figure 13 b). We call this operation *anellini compression* with  $D_\ell$  the *anellini disc*. Note that  $S_1^3 \cap D \subset D_\ell$  and  $|D_\ell \cap D| = 1$  where  $D_\ell$  is viewed as a disc. If instead we had  $D_1 \subset X_0^S$ , then  $S^3$  is isotopic to a presentation with  $D_1$  carved and with embedded surgery along  $D_\ell$  with again  $D_\ell$  the anellini disc. In both cases, carving and embedded surgery are from the point of view of  $X_0^S$ .

Figure 13 c) shows the case when  $D \cap S^3 = k_1 \cup k_2$ . Here  $S^3$  is isotopic to the presentation of that figure, where  $\ell_1, \ell_2$  are the linking circles. Note that a dot (resp. circle) corresponds to a carving along a disc whose boundary germ points into  $X_0^S$  (resp.  $X_0^N$ ). While the induced framings on linking circles are always 0-framed, the induced framings on the knots are not necessarily 0-framed even though they might be labelled with a 0. In our case  $S^3$  is first carved along the disc  $D_1$  and then embeddedly surgered along the disc  $D_2$ . It is also carved along the standard disc  $D_{\ell_2}$  followed by an embedded surgery along the disc  $D_{\ell_1}$ . Here  $D_{\ell_1}$  consists of a tube that starts at  $k_1$  and follows an arc in  $D_2$  that is capped off with a copy of  $D_{\ell_2}$ . We say that  $k_2$  *nests* in  $k_1$  and  $\ell_1$  *nests* in  $\ell_2$ , i.e. it enters the hollow created by the carving or embedded surgery. Nesting is combinatorially recoded as in Figure 13 d). If  $\partial D \cup D_1 \subset X_0^N$ , then we have the similar situation with circles replaced by dots and vice versa.

In general a disc  $D$  may intersect  $S^3$  in a finite set  $k_1, \dots, k_n$  of simple closed curves. Here  $S^3$  is isotopic to  $S_1^3$  which is represented by a set of carvings and embedded surgeries along the  $k_i$ 's and their linking circles and  $S_1^3$  intersects  $D$  only along its anellini discs. Figure 14 b) shows a *nesting diagram* that schematically indicates the carvings and surgeries for the disc  $D$  of Figure 14 a) with  $\partial D \subset \text{int}(X_0^S)$ . The boxes indicate that the knots can be knotted and linked with each other.

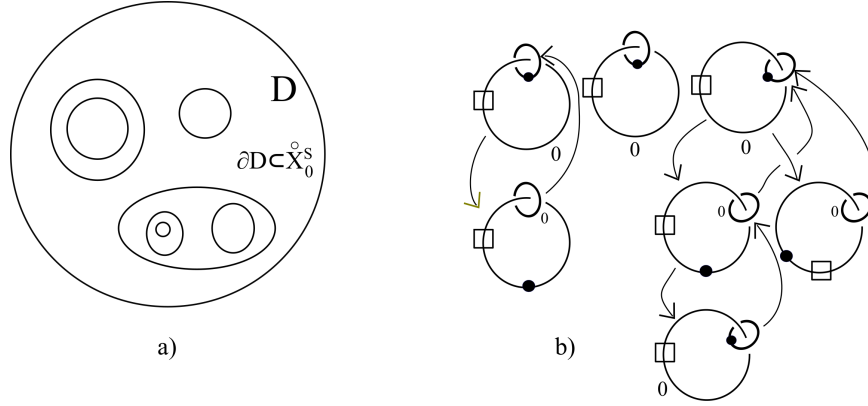


FIGURE 14. The Nesting Partial Order

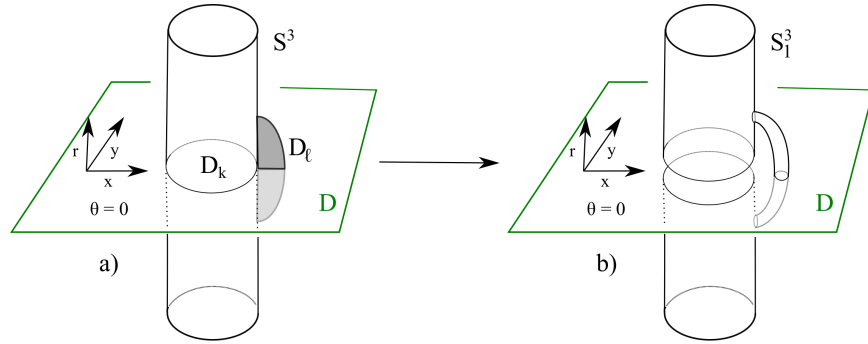


FIGURE 15. Anellini Compression: The D view

**Remark 10.8.** Figure 15 shows the anellini compression from the point of view of the disc  $D$ . Figure 15 a) shows  $D_k$  and  $D_\ell$  and Figure 15 shows the result of the compression. The  $D_\ell$  is contained in the  $y = 0$  plane and the shading indicates the projection to the  $y = 0, \theta = 0$  plane. Using cylindrical  $(x, y, r, \theta)$  local coordinates we can assume that  $D, D_k, D_\ell$ , and  $S^3$  are all locally invariant under  $\theta$  rotation and hence so is the anellini compression viewed locally.

To complete Step 3, we now show how to glissade from the point of view of anellini discs.

**Lemma 10.9.** *If  $S'_{5/8} \cap N(\widetilde{\mathcal{G}}^{\text{std}})$  is contained in anellini discs  $A'_{5/8}$  and the intersection with these discs is a union of fibers of  $N(\widetilde{\mathcal{G}}^{\text{std}})$ , then viewed in  $\tilde{V}$  via  $\tilde{\lambda}$  these anellini discs  $A_{5/8}$  have open subdiscs removed, one for each component of  $A'_{5/8} \cap N(\widetilde{\mathcal{G}}^{\text{std}})$  and their boundaries are a union of sections of  $N^1(\tilde{\Omega})$ . The glissade creates the anellini discs  $A_1$  obtained by capping off the boundary circles of  $A_{5/8}$  with annuli in  $N(\tilde{\Omega})$  together with copies of  $\tilde{\mathcal{R}}^0_{5/8}$ . Equivalently, first obtain  $A'_1$  from  $A'_{5/8}$  by tubing off each intersection with  $\widetilde{\mathcal{G}}^{\text{std}}$  with a parallel copy of a component of  $\tilde{\mathcal{R}}_{5/8}$  and then let  $A_1 = \lambda(A'_1)$ .  $\square$*

**Step 4:** Change our perspective of  $N(\tilde{\mathcal{D}})$  to pies with fillings and crust.

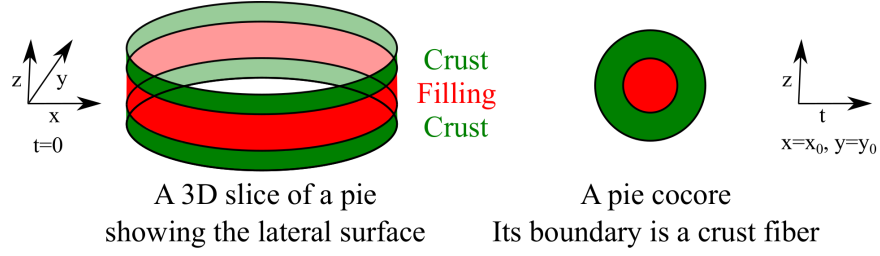


FIGURE 16. A Pie

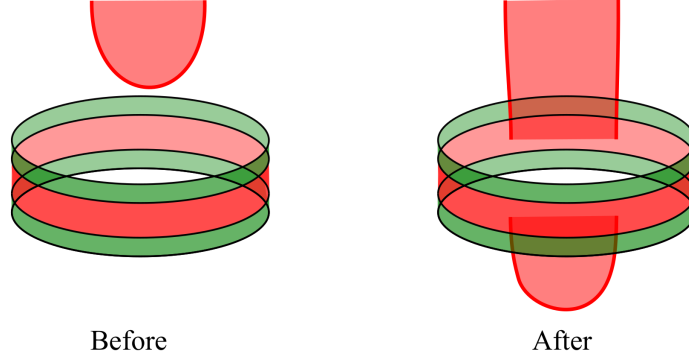


FIGURE 17. Finger Move: Pie View

**Definition 10.10.** Up to smoothing corners we will view the component of  $N(\tilde{\Omega} \cup \tilde{R}_{1/4}^0)$  associated to  $\tilde{w}_j$  as the thin *pie*  $D_j^2 \times D_\epsilon^2 := P_j$  where the *filling*  $N(\tilde{R}_{j,1/4}^0)$  corresponds to  $D_j^2 \times \frac{1}{2}D_\epsilon^2$  and the *crust*  $N(\tilde{w}_j)$  corresponds to  $D_j^2 \times (D_\epsilon^2 \setminus \text{int}(\frac{1}{2}D_\epsilon^2))$ . From this point of view we equate fibers of  $N^1(\tilde{G}^{\text{std}}_j) \setminus \text{int}(N(\tilde{R}^{\text{std}}_j))$  called *crust fibers* with circles  $x \times \partial D_\epsilon^2$ ,  $x \in D_j^2$  and parallel copies of  $\tilde{R}_{j,1/4}^0$  with the discs  $x \times \frac{1}{2}D_\epsilon^2$ ,  $x \in \partial D_j^2$ . Discs of the form  $D^2 \times (z_0, t_0)$  where  $(z_0, t_0) \subset \frac{1}{2}D_\epsilon$  are called *sections of the filling*. Discs of the form  $(x_0, y_0) \times D_\epsilon^2$  are called *pie cocores*. See Figure 16 a) which shows  $D^2 \times [-\epsilon, \epsilon]$ , where  $[-\epsilon, \epsilon]$  is a diameter of  $D^2_\epsilon$  and Figure 16 b) which shows a pie cocore.

**Remark 10.11.** *Finger moves from the pie view:* See Figure 17. Note that a finger move in  $\tilde{V}$  can be now viewed as a regular homotopy. While the points inside the pie are not really there and indeed not shown in the figure, this point of view when completed will define an explicit regular homotopy from  $\mathbb{S}_0$  to  $\mathbb{S}_1$ . Note that solid lines denote the part of  $\mathcal{R}_t^0$  in the present, while the shading denotes the parts in the past or the future.

**Remark 10.12.** *The glissade from the pie view:* Since  $\tilde{R}_{j,5/8}^0$  coincides with  $\tilde{R}_{j,1/4}^0$  near  $\partial \tilde{R}_{j,1/4}^0$  we can assume that  $N(\tilde{R}_{j,5/8}^0) := D_j^2 \times \tilde{R}_{j,5/8}^0$  and fiberwise coincides with  $N(\tilde{R}_{j,1/4}^0) := D_j^2 \times \frac{1}{2}D_\epsilon^2$  near  $D_j^2 \times \partial \frac{1}{2}D_\epsilon^2$ . If  $A_{5/8}$  are anellini discs that intersect  $D_j^2 \times D_\epsilon^2$  in  $K_j \times \partial D_\epsilon^2$ ,

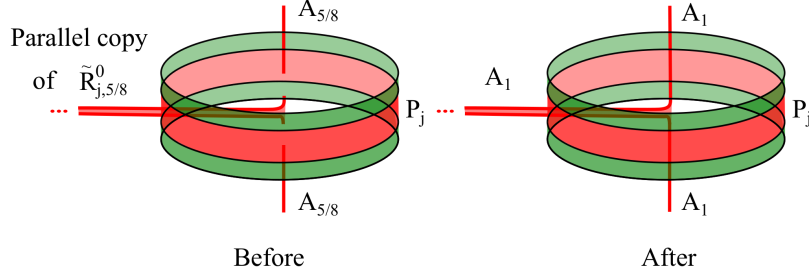


FIGURE 18. The Glissade: Pie View

then the glissade gives

$$A_1 = A_{5/8} \cup_j (K_j \times (D_\epsilon^2 \setminus \text{int } \frac{1}{2}D_\epsilon^2)) \cup_j (K_j \times \tilde{R}_{j,5/8}^0).$$

See Figure 18. Here  $A_{5/8}$  intersects  $P_j$  in a single crust fiber. The solid lines denote that part of  $\tilde{R}_{j,5/8}^0$  in the present, while the shading denotes parts in the past or the future. Also,  $\tilde{R}_{j,5/8}^0$  and more generally  $\tilde{\mathcal{R}}_{5/8}^0$ , may intersect  $P_j$  in sections of the filling. These are not shown in the figure. This completes Step 4.

**Step 5:** Construct  $\mathbb{S}_1$  from the pie perspective using anellini compressions.

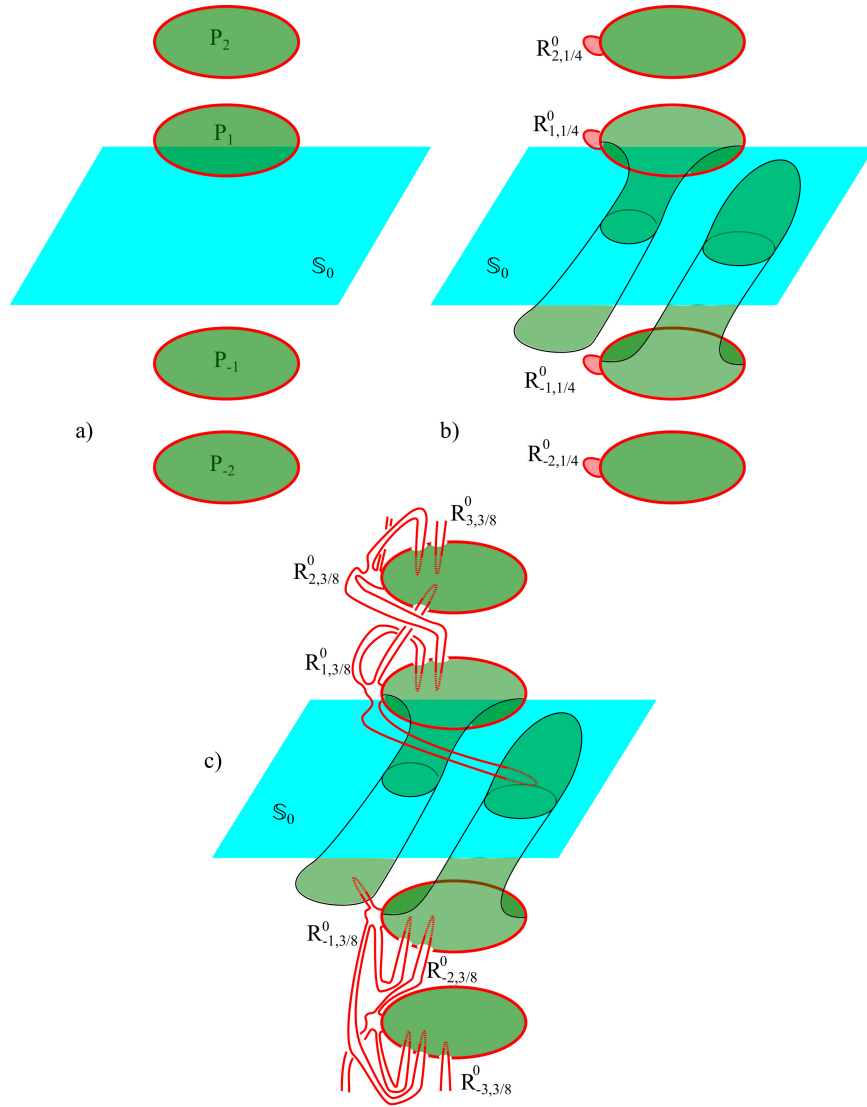
In this section we go through the construction of Step 2 using the technology developed in Steps 3 and 4. We also record as *data memos* the nesting and intersection data needed for our  $F|W$ -carving/surgery presentation. In particular, we need to keep track of iterated nesting, i.e. whether a nests d directly or through intermediate nestings or both.

i) *Prepare for the finger moves:* For each  $j > 0$  (resp.  $j < 0$ ) for which there is an arm from  $\widetilde{R}_i^{\text{std}}$  to  $\widetilde{G}_j^{\text{std}}$  with  $i < 0$  (resp.  $i > 0$ ), do a finger move from  $P_j$  into  $\mathbb{S}_{1/4} = \mathbb{S}_0$ . See Figures 19 a), b). This is done away from where  $\mathcal{R}_{1/4}^0$  is located. In Figure b)  $\mathcal{R}_{1/4}^0$  has been isotoped to pop slightly outside of  $P_j$ . The dark lines indicated the present and the shading the projection to the present.

**Data Memo 10.13.** a) Viewed in  $\mathbb{S}_0$ ,  $P_j \cap \mathbb{S}_{1/4}$  is an unknotted solid torus  $N^3(g_j)$ . The knots  $g_j$  arising from this preparatory move are unknotted and unlinked in  $\mathbb{S}_0$ .

b) Call a pie *plunged* if it intersects  $\mathbb{S}_0$ . Let  $P_{n_1}, \dots, P_{n_p}$  denote the plunged pies with  $n_i > 0$  and  $P_{s_1}, \dots, P_{s_q}$  those with  $s_j < 0$  and let  $g_{n_1}, \dots, g_{n_p}, g_{s_1}, \dots, g_{s_q}$  denote the corresponding knots.

c) If  $j < 0$  (resp.  $j > 0$ ), then  $P_j \cap X_0^N = \sigma_j$  (resp.  $P_j \cap X_0^S$ ) is the product of a standard 2-disc  $D_{g_j}$  with pie cocores, where  $\partial D_{g_j} = g_j$ . In particular, those 2-discs respectively in  $X_0^N$  and  $X_0^S$  are unknotted and unlinked.

FIGURE 19. Finger move in  $\tilde{V}$ : Pie view

ii) *Do the finger moves*: Figure 19 c) shows  $\mathcal{R}_{3/8}^0$  after finger moves of  $R_{i,1/4}^0$  into  $P_{i-1}, P_i, P_{i+1}$ ,  $i \in \mathbb{Z}$ , where indices need to be adjusted as in Notation 10.3 when one of them equals 0. The figure only shows the intersection of  $\mathcal{R}_{3/8}^0$  with the present.

iii) *Prepare for the Whitney moves*: This requires first doing anellini compressions and then moving the anellini discs off of their Whitney discs.

Let  $\tilde{w}_{ij}$  denote a Whitney disc from  $\tilde{R}_{i,1/2}$  to  $\tilde{G}_{j}^{\text{std}}$  and  $w_{ij} = \lambda(\tilde{w}_{ij}) \subset \tilde{V}$ . The number of such discs is the number of fingers in the corresponding hand and  $\cup w_{ij}$  will denote the union of such discs. For each  $w_{ij}$  do anellini compressions to modify  $\mathbb{S}_{1/2}$  to  $\mathbb{S}_{17/32}$ . Figure 7 shows the case of a hand having two fingers.

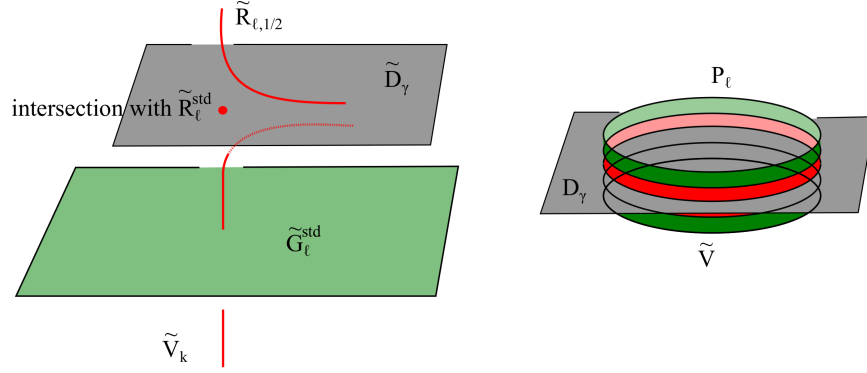


FIGURE 20. A Section of a Pie Filling arising from an intersection with  $\widetilde{R}_{t,\ell}^{\text{std}}$

**Data Memo 10.14.** a) Each  $w_{ij}$  gives rise to a nesting diagram as in Figure 7 where a) (resp. b)) shows the case where  $i < 0$  (resp.  $i > 0$ ). The diagram should be viewed as lying in  $\mathbb{S}_0$ .

b)  $\mathbb{S}_{17/32} \cap w_{ij}$  is contained in the anellini discs which span the linking circles of the knots of the diagram. These discs induce 0-framings on the linking circles.

c) If  $\gamma$  is a component of  $\mathbb{S}_{1/2} \cap w_{ij}$ , then let  $D_\gamma \subset w_{ij}$  denote the disc bounded by  $\gamma$  and  $\tilde{D}_\gamma = \lambda^{-1}(D_\gamma)$ . Such a  $\gamma$  corresponds to a knot  $K_\gamma \subset \mathbb{S}_0$  in the nesting diagram and comes with a linking circle  $L_\gamma$ . Let  $D_{ij} := \cup_\gamma D_\gamma \subset (\cup w_{ij})$  and  $\tilde{D}_{ij} = \lambda^{-1}(D_{ij})$ . Give the  $\gamma$ 's the induced partial order coming from the nesting diagram knots, i.e. ordered by inclusion. Let  $D_{ij}^\gamma = D_\gamma \setminus \cup_{\beta < \gamma} \text{int}(D_\beta)$  and  $\tilde{D}_{ij}^\gamma = \lambda^{-1}(D_{ij}^\gamma)$ .

d) Each point of  $\tilde{D}_{ij}^\gamma \cap \widetilde{R}_{t,\ell}^{\text{std}}$  gives rise to a section  $\subset D_{ij}^\gamma \cap P_\ell$  of  $P_\ell$ 's filling. See Figure 20.

e) Each point of  $(\cup \tilde{w}_{ij} \setminus \tilde{D}_{ij}) \cap \widetilde{R}_{t,\ell}^{\text{std}}$  gives rise to a section  $\subset (\cup w_{ij} \setminus D_{ij}) \cap P_\ell$  of  $P_\ell$ 's filling.

f)  $\cup w_{ij}$  nests once in each  $K_\gamma$  for outermost  $\gamma \subset \cup w_{ij}$ .

g) We call the knots arising from these anellini compressions *southern knots* (resp. *northern knots*) if  $i < 0$  (resp.  $i > 0$ ). A given southern (resp. northern) knot is generically denoted  $S_{ij}$  (resp.  $N_{ij}$ ) if it arises from a  $w_{ij}$  Whitney disc. When we say a knot (resp. linking circle) is *northern maximal* or *minimal* we mean that it is maximal or minimal among the northern knots (resp. linking circles). Similarly we may refer to *southern minimal* etc. knots or linking circles.

Next isotope the northern and southern minimal anellini discs off of the Whitney discs to obtain  $\mathbb{S}_{5/8}$  at the cost of creating two oppositely signed intersections with  $P_j$  for each such anellini disc.

**Data Memo 10.15.** a) Here we are viewing  $P_j$  as a 2-disc. In reality, each northern or southern minimal anellini disc intersects exactly one  $P_j$  in two pie cocores. Note that to follow Step 2 we need to remove the interior of these cocores. The advantage of keeping them is that we can continue to work with the anellini discs as discs. In particular, to keep track of the discs spanning the linking circles. Also, we continue to extend the modification of  $\mathbb{S}_0$  to  $\mathbb{S}_1$  as a regular homotopy.

b) Since an isotopy of an anellini disc extends to an isotopy of those discs nested inside, the above isotopy extends to all the anellini discs intersecting  $w_{ij}$  and hence moves all of them off of  $w_{ij}$ .

c) If  $i < 0$  (resp.  $i > 0$ ) and  $j > 0$  (resp.  $j < 0$ ), then all these intersections  $\subset \sigma_j$  otherwise they are disjoint from all the  $\sigma_j$ 's.

iv) *Do the Whitney moves:*

**Data Memo 10.16.** a)  $R_{i,5/8}^0$  nests twice with opposite sign through the northern and southern maximal knots of the form  $S_{ij}$  or  $N_{ij}$ , some  $j$ .

b) Each point of  $(\cup \tilde{w}_{ij} \setminus \tilde{D}_{ij}) \cap \widetilde{R}^{\text{std}}_\ell$  induces two filling sections of  $P_\ell \cap R_{i,5/8}^0$ .

v) *Do the glissade:* First prepare for the glissade by doing anellini compressions to the  $g_j$ 's along the discs  $D_j$  in the plunged pies. Let  $\mathbb{S}_{21/32}$  denote the isotoped  $\mathbb{S}_{5/8}$ . Second, do the glissade as in Remark 10.12 to obtain  $\mathbb{S}_1$ . See Figures 21 a), b). It depicts the anellini compression along  $D_j$  from the  $P_j$  point of view, i.e. here  $P_j$  looks flat rather than plunged. In this figure only the intersections with the present are shown. Figure 21 a) depicts two anellini discs while b) includes  $D_{g'_j}$ , a third.

**Data Memo 10.17.** a) After the compressions a  $P_j$  only intersects  $\mathbb{S}_{21/32}$  in anellini discs. This preparatory operation requires isotoping those anellini discs that intersect  $D_j$ .

b)  $R_{i,5/8}^0$  will nest  $g_\ell$  only if  $(\cup \tilde{w}_{ij} \setminus (\cup_j \tilde{D}_{ij})) \cap \widetilde{R}^{\text{std}}_\ell \neq \emptyset$  and the knot  $K_\gamma, \gamma \subset w_{ij}$  will nest  $g_\ell$  only if  $\tilde{D}_{ij}^\gamma \cap \widetilde{R}^{\text{std}}_\ell \neq \emptyset$ .

c) If  $j > 0$  (resp.  $j < 0$ ), then the linking circle  $g'_j$  lodges exactly those southern (resp. northern) minimal linking circles  $S'_{ij}$  where  $i < 0$  (resp.  $i > 0$ ).

d) The glissade involves modifying an anellini disc  $A$  by replacing its intersection with  $P_j$ , which are pie cocores, with parallel copies of  $R_{j,5/8}^0$ .

e) A pie cocore shares its boundary with a parallel copy of  $R_{j,5/8}^0$ . It follows by Lemma 10.4 that it is isotopic to  $R_{j,5/8}^0$  fixing a neighborhood of its boundary pointwise, It follows that the glissade does not change the induced framings of the boundary of the anellini discs.

f) With  $\mathbb{S}_{21/32}$  viewed as an immersed 3-sphere, it follows that the glissade can be realized as a regular homotopy.

g) Following the glissade,  $g'_j$  will nest exactly those knots nested by  $R_{j,5/8}^0$ .

h) If  $i < 0$  and  $j > 0$ , then a southern minimal linking circle  $S'_{ij}$  will only nest  $g'_j$ . If instead  $j < 0$ , then  $S'_{ij}$  will nest exactly those southern maximal knots and  $g_\ell$ 's nested by  $R_{j,5/8}^0$ .

i) If  $i > 0$  and  $j < 0$ , then a northern minimal linking circle  $N'_{ij}$  will only nest  $g'_j$ . If instead  $j < 0$ , then  $N'_{ij}$  will nest exactly those northern maximal knots and  $g_\ell$ 's nested by  $R_{j,5/8}^0$ .

This completes Step 5.

**Step 6:** Organize the data. By construction  $\mathbb{S}_1$  is obtained from  $\mathbb{S}_0$  by a carving/surgery presentation. The various data memos record the details which we summarize now and show that we have a  $F|W$ -carving/surgery presentation.

The passage of  $\mathbb{S}_0$  to  $\mathbb{S}_1$  involved modifications of  $\mathbb{S}_0$  while: A) preparing for the Whitney moves, B) preparing for the glissade and C) doing the glissade. The Whitney preparatory



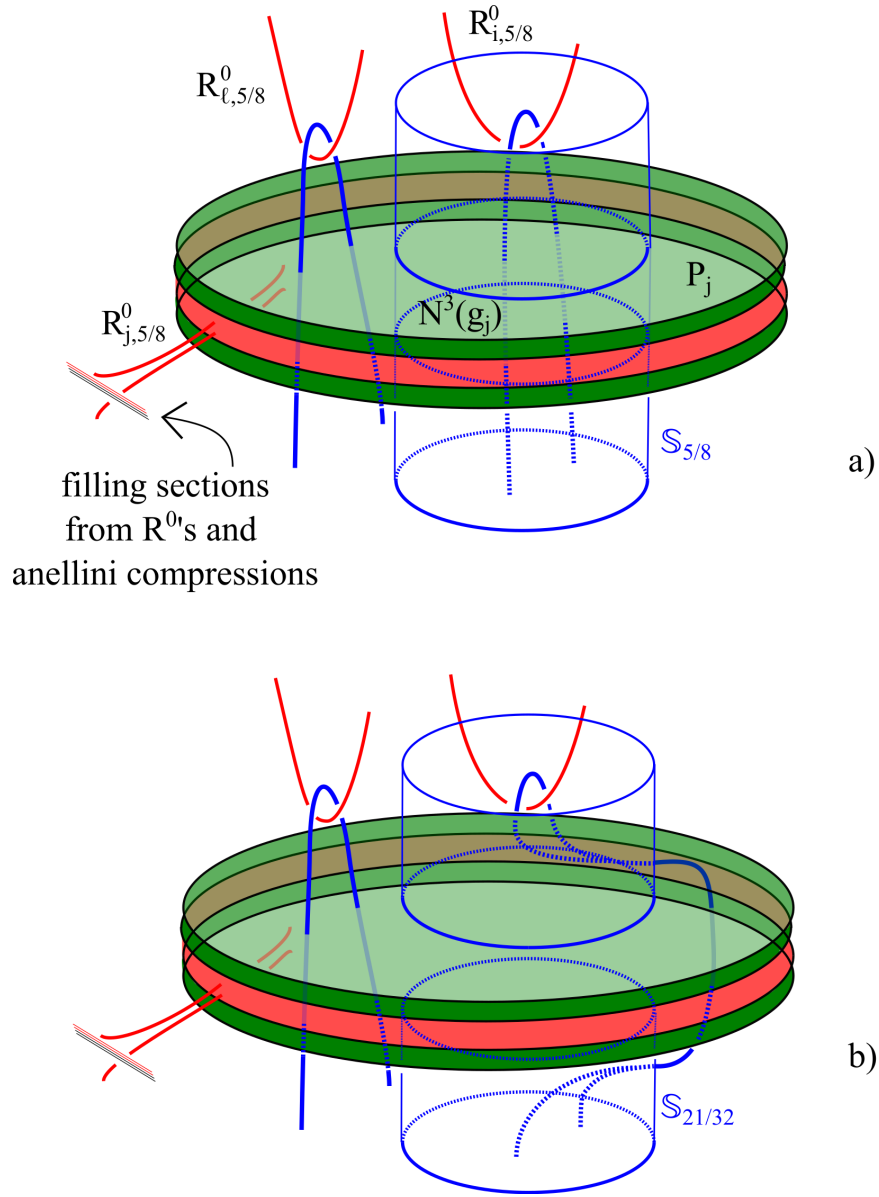


FIGURE 21. An Anellini compression after the Whitney moves

moves create the northern and southern knots and their linking circles. Each northern or southern knot  $K$  corresponds to a  $\gamma \subset w_{ij}$  and by construction  $D_K = D_\gamma$ . Thus the nesting relations among the northern or southern knots arise during A) and give the relations described in Definition 9.6 1b) and 1c). The  $g_j$ 's were defined in Data Memo 10.13 as were their spanning discs  $D_{g_j}$ , though they weren't created until B). By construction these  $g_j$ 's are unknotted and unlinked in  $\mathbb{S}_0$  and the intersections of the  $D_{g_j}$ 's with either  $X_0^N$  or  $X_0^S$  are also unknotted and unlinked. A  $D_K$  will nest a  $D_{g_j}$  only if it intersects  $P_j$  in filling sections which occurs only if  $\tilde{D}_\gamma \cap \widetilde{R}^{\text{std}}_j \neq \emptyset$ , where  $K = K_\gamma$ . See Data Memo's 10.14 d). These

nestings give a possibly proper subset of the nestings described in the second and fourth paragraphs of Definition 9.6 2a).

The spanning disc  $D_{K'}$  of the linking circle  $K'$  of a southern or northern knot  $K$  is constructed in 3 steps. To start with we have  $D_{K'}^1$ , the standard spanning disc constructed in the annallini compression operation during A). The northern and southern minimal  $D_{K'}^1$ 's are isotoped during A) which induces the isotopy of the others. Let  $D_{K'}^2$  denote the isotoped  $D_{K'}^1$ . Note that it intersects some  $P_j$  in two pie cocores of opposite sign. These pie cocores are then replaced by copies of  $R_{j,5/8}^0$  to obtain  $D_{K'}$ . By construction  $D_{K'}^1$  induces the 0-framing on  $K'$ . Isotopy does not change that framing nor does the pie cocore replacement by Data Memo 10.17 e). Similarly the spanning disc  $D_{g'_j}$  is first obtained by the anellini compression operation to obtain  $D_{g'_j}^1$  which intersects  $P_j$  in a single pie cocore.  $D_{g'_j}$  is obtained by replacing this cocore with a copy of  $R_{j,5/8}^0$ . The previous argument shows that it induces the 0-framing on  $g'_j$ . By construction no spanning disc of a knot nests in the spanning disc of a linking circle. Nestings of the  $D_{K'}^1$ 's induce the nestings of Definition 9.6 1b) and 1c) and these nestings remain during the passages to the  $D_{K'}$ 's. The nestings of the first and third paragraphs of Definition 9.6 2a) arise during B), see Data Memo 10.17 c). The nestings created during C) are a possibly proper subset of those described in Definition 9.6 a), 2b) and 2c), see Data Memo 10.17 h) and i). This completes of Step 6 and hence the proof of Theorem 10.1.  $\square$

As noted all steps in the transformation of  $\mathbb{S}_0$  to  $\mathbb{S}_1$  can be achieved through regular homotopy and hence the following result.

**Theorem 10.18.** *Any smooth 3-sphere in  $S^4$  is regularly homotopic to the standard 3-sphere.*  $\square$

## 11. UPGRADING TO AN OPTIMIZED PRESENTATION

In this section all  $F|W$  systems are  $\partial$ -germ coinciding.

**Construction 11.1.** With notation as in 3.7, given the  $F|W$  system  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  with  $\mathcal{R}$  in AHF form, then let  $\mathbb{S}'$  denote the 3-sphere  $\pi^{-1}(1/2) \subset V_k$ . By passing to a finite cover we can assume that  $\text{Max}_{i,j} \{\text{Diam}(\pi(R_i)), \text{Diam}(\pi(w_j))\} < k/100$  for all  $R_i \in \mathcal{R}$  and  $w_j \in \mathcal{W}$ . Construct  $\hat{\mathbb{S}}$  as follows. If  $R_i \cap G_j \neq \emptyset$ , where  $i \in [1, k/100]$  and  $j \in [0, -k/100]$  (resp.  $i \in [0, -k/100]$  and  $j \in [1, k/100]$ ), then do a finger move of  $\mathbb{S}'$  into  $G_j$ . Our  $\hat{\mathbb{S}}$  is the result of these finger moves. Let  $\hat{B}_K$  denote  $\hat{\mathbb{S}} \cap \mathcal{G}$ .

**Remarks 11.2.** i) There is an ambient isotopy  $F_t$  of  $V_k$  such that  $F_1(\mathbb{S}') = \hat{\mathbb{S}}$  and  $F_1(\mathcal{R}^{\text{std}}) = \mathcal{R}$ .

ii) After passing to  $\tilde{V}_k$ ,  $\hat{\mathbb{S}}$  will become our  $\mathbb{S}'_{3/8}$  in the proof of Theorem 10.1.

**Definition 11.3.** The  $F|W$  system  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is  $\hat{\mathbb{S}}$  adapted if  $\hat{\mathbb{S}}$  arises from Construction 11.1 and for every  $w_i \in \mathcal{W}$ ,  $w_i \cap \hat{\mathbb{S}}$  is a union of simple closed curves  $\alpha_{i_1}, \dots, \alpha_{i_p}$  such that

- i) For all  $j$ ,  $\text{level}(\alpha_{i_j}) \leq 2$ , where level is calculated within  $w_i$ .
- ii) If  $D_{i_j} \subset w_i$  is the disc bounded by  $\alpha_{i_j}$ , then  $D_{i_j}$  induces the 0-framing on  $\alpha_{i_j}$ .
- iii) The  $\alpha_{i_j}$ 's are the disjoint union of  $A \cup B$  where  $A \cup \hat{B}_L$  and  $B \cup \hat{B}_L$  are unlinks in  $\hat{\mathbb{S}}$ .
- iv)  $B$  includes all the level-2 curves and if  $\alpha_{i_j} \subset B$ , then  $D_{i_j} \cap \mathcal{R}^{\text{std}} = \emptyset$ .

**Lemma 11.4.** *If  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  satisfies the conclusion of Proposition 8.2 and together with  $\hat{\mathbb{S}}$  are as in Construction 11.1, then  $\mathcal{W}$  can be isotoped to be  $\hat{\mathbb{S}}$ -adapted, via an isotopy supported away from  $\mathcal{R} \cup \mathcal{G}$ .*

*Proof.* We first show that if  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  is  $\partial$ -germ coinciding, then  $\mathcal{W}$  can be isotoped to satisfy

- i') For all  $j$ , level  $(\alpha_{i_j}) \leq 2$ , where level is calculated within  $w_i$ .
- ii') If level  $(\alpha_{i_j}) = 2$  and  $D_{i_j} \subset w_i$  is the disc bounded by  $\alpha_{i_j}$ , then  $D_{i_j} \cap \mathcal{R}^{\text{std}} = \emptyset$ .
- iii') The level-2 curves  $\alpha_{i_j}$  bound pairwise disjoint embedded discs  $F_{i_j} \subset \hat{\mathbb{S}} \setminus \mathcal{G}$ .

The idea is to attempt to do  $\partial$ -compressions to  $\mathcal{W}$  along pairwise disjoint embedded discs to reduce  $\mathcal{W} \cap \hat{\mathbb{S}}$  to a collection of level-1 curves. In practice, interiors of these discs may intersect  $\mathcal{W}$ . Isotoping these intersections away will create embedded discs at the cost of new intersections of  $\mathcal{W}$  with  $\hat{\mathbb{S}}$ . Some of them will be level-2, however their intersections will satisfy ii') and iii').

Let  $O(\mathcal{W}) \subset \mathcal{W}$  consist of  $\mathcal{W} \cap \hat{\mathbb{S}}$  together with those points separated from  $\partial\mathcal{W}$  by an odd number of components of  $\mathcal{W} \cap \hat{\mathbb{S}}$ . Let  $\beta_1, \dots, \beta_n \subset O(\mathcal{W})$  denote pairwise disjoint embedded arcs that cut  $O(\mathcal{W})$  into discs. Let  $E_1, \dots, E_n$  be immersed discs such that  $\partial E_i = \beta_r \cup \gamma_r$  where  $\gamma_r \subset \hat{\mathbb{S}}$ ,  $\text{int}(E_i) \cap \hat{\mathbb{S}} = \emptyset$  and  $\pi(E_i) \subset [-k/20, k/20]$ . These discs might not be suitable because they may a) intersect  $\mathcal{G}$ , b) intersect  $\mathcal{R}$ , c) intersect each other, d) have the wrong framing, and e) intersect  $\mathcal{W}$  in their interiors. By rechoosing  $\gamma_r$  so that a pushoff of the loop  $\gamma_r \cup \beta_r$  is null homotopic in  $V_k \setminus (\hat{\mathbb{S}} \cup \mathcal{G})$  we address a). If  $x \in E_i \cap R_j$ , then let  $R'_j$  be a parallel copy of  $G_j$  that intersects  $\mathcal{R}$  once and is disjoint from  $\hat{\mathbb{S}}$ . Tube off a neighborhood of  $x \in E_i$  with a copy of  $R'_j$  using an arc from  $x$  to  $R_j \cap R'_j$ . This will likely create new intersections with  $\mathcal{W}$  and the  $E_j$ 's. To address c) do finger moves to the  $E_i$ 's to make them embedded at the cost of creating new intersections with  $\mathcal{W}$ . To address d) do  $\partial$ -twisting as in [FQ] (see also [E]). Again the cost is new intersections with  $\mathcal{W}$ . Finally for each  $x \in \text{int}(E_i) \cap \mathcal{W}$  choose pairwise disjoint arcs  $\delta_x \subset E_i \setminus (\beta_i \cup \mathcal{R}^{\text{std}})$  from  $x$  to  $\hat{\mathbb{S}}$ . Do finger moves to  $\mathcal{W}$  along the  $\delta_x$ 's and then a bit further to make  $(\cup \text{int}(E_i)) \cap \mathcal{W} = \emptyset$ . Denote the new components of  $\mathcal{W} \cap \hat{\mathbb{S}}$  corresponding to  $x$  by  $\epsilon_x$ . Here we abused notation by having  $\mathcal{W}$  denote the modified  $\mathcal{W}$ . Note that each component of  $\mathcal{W} \cap \hat{\mathbb{S}}$  is now level- $\leq 2$  and the level-2 ones are among the  $\epsilon_x$ 's. By construction each  $\epsilon_x$  bounds a disc  $D_x \subset \mathcal{W}$  disjoint from  $\mathcal{R}^{\text{std}}$  and the  $\epsilon_x$ 's bound pairwise disjoint discs in  $\hat{\mathbb{S}}$  disjoint from  $\mathcal{G}$ .

Now assume that  $(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  satisfies the conclusion of Proposition 8.2. We now modify the previous argument to isotope the plates so that their intersections with  $\hat{\mathbb{S}}$  satisfy i'), ii') and iii'), where level (resp. subdisc) is calculated (resp. constructed) within the plates. We also have iv'): the  $D_x$ 's are disjoint from the concordance 1-handles. To start with if  $P$  is a plate, then define  $O(P)$  analogously as above to  $O(\mathcal{W})$ . Next choose  $\beta_i$ 's as above to be disjoint from  $(\text{int}(1\text{-handles}) \cap \text{plates})$  and then choose  $E_i$ 's as above to be disjoint from  $\mathcal{G}$ . These  $E_i$ 's may intersect the bases and the beams, but such intersections can be eliminated at the cost of new intersections with  $\mathcal{R}$ . Next eliminate as above all the intersections of the  $E_i$ 's with  $\mathcal{R}$  at the cost of creating new intersections among the  $E_i$ 's and  $E_i$ 's with the plates. Similarly eliminate the  $E_i/E_j$  intersections at the cost of creating new intersections of the  $E_i$ 's and the plates. We can assume that the plate/plate intersections are disjoint from the 1-handles, although the  $E_i$ 's themselves may intersect them. Now complete the argument

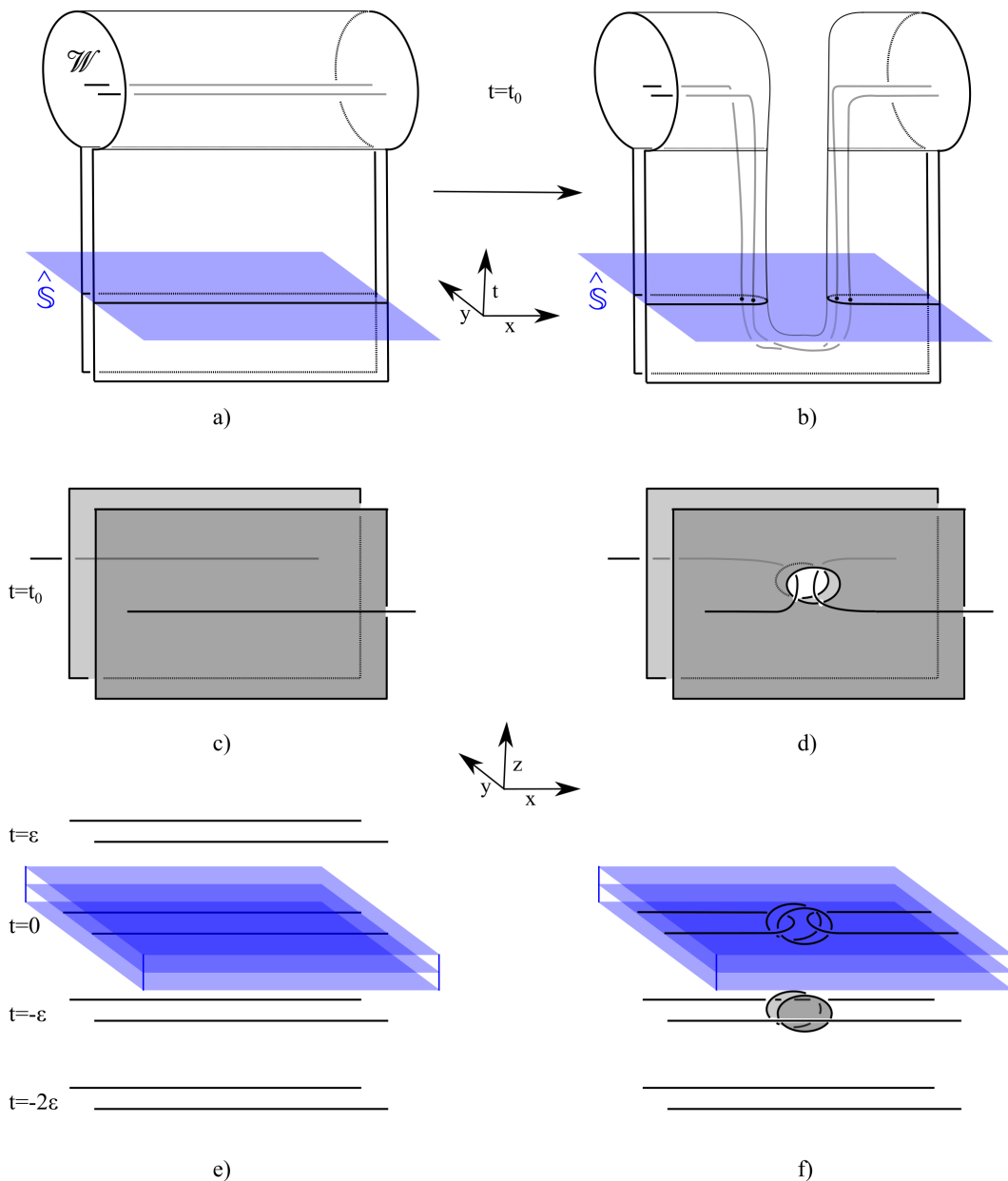


FIGURE 22. A beam compression

as in the previous paragraph. Note that the  $D_x$ 's lie very close to  $\hat{S}$  and hence are disjoint from the 1-handles.

For each beam let  $D_b$  be a cocore of the corresponding 3-dimensional 2-handle. For each component  $Q$  of  $O(P)$  let  $\gamma_q \subset P$  be a path disjoint from  $\mathcal{R}^{\text{std}}$  from  $Q$  to a parallel copy  $D_q$  of  $D_b$ . Note that  $Q$  is a planar surface corresponding to two parallel copies  $Q', Q'' \subset \mathcal{W}$ . Let  $\partial_e Q$  denote the component of  $\partial Q$  intersecting  $\gamma_q$  and  $D_Q$  the disc in  $P$  bounded by  $\partial_e Q$ . Similarly define  $\partial_e Q', \partial_e Q'', D_{Q'}$  and  $D_{Q''}$ . Now for each  $Q$ , do boundary compressions of  $\mathcal{W}$  into  $\hat{S}$  following  $D_q \cup \gamma_q$ , simultaneously moving neighborhoods of  $\text{int } D_q \cap \mathcal{W}$  out of the way,

thereby creating new intersections of  $\mathcal{W}$  with  $\hat{\mathbb{S}}$  that are also denoted  $\epsilon_x$ 's. The boundary compression bands together  $Q'$  and  $Q''$  to obtain  $Q^*$  where the band connects  $\partial_e Q'$  to  $\partial_e Q''$ . Denote by  $\partial_e Q^*$  its new *exterior boundary* and  $D_{Q^*}$  its new disc. See Figure 22. Figures a) and b), c) and d), e) and f) show various 3-dimensional slices before and after the boundary compression. The resulting  $\mathcal{W}$  satisfies the conclusion of the lemma. Note that all the  $\partial_e Q^*$ 's are level-1 and the new and old  $\epsilon_x$ 's are level- $\leq 2$ . Here  $B$  is the union of the  $\epsilon_x$ 's which by construction satisfy iii) and iv) and  $A$  is the union of the  $\partial_e Q^*$ 's which together with  $B_L$  is also an unlink. Indeed,  $\partial_e Q' \cup \partial_e Q''$  bound a thin annulus  $\subset \hat{\mathbb{S}}$  disjoint from  $\hat{B}_L$  and  $\partial_e Q^* = \partial E_{Q^*} \subset \hat{\mathbb{S}}$  where  $E_{Q^*}$  is the result of cutting this annulus along an arc. Since there is a regular homotopy of  $D_{Q^*}$  to  $E_{Q^*}$  fixing the boundary pointwise,  $D_{Q^*}$  induces the 0-framing on  $\partial_e Q^*$ .  $\square$

**Corollary 11.5.** *If  $\phi \in \text{Diff}_0(S^1 \times S^3)$ , then  $\phi$  is  $S$ -equivalent to  $\phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$  where the  $F|W$  system satisfies the conclusion of Proposition 8.2 and is  $\hat{\mathbb{S}}$ -adapted.*  $\square$

**Theorem 11.6.** *If  $\mathbb{S}$  is a smooth 3-sphere in  $S^4$ , then  $\mathbb{S}$  has an optimized FWCS presentation.*

*Proof.* Apply the proof of Theorem 10.1 to a  $\hat{\mathbb{S}}$  adapted  $F|W$  system. Condition i) of Definition 11.3 follows from i) of Lemma 11.4. The proof of Theorem 10.1 shows that if  $k \in B_L$ , then  $D_k$  is a standard disc and hence  $k$  is 0-framed, otherwise the framing is induced by  $D_k$  the disc in  $\mathcal{W}$  bounded by  $k$  and hence Condition ii) follows from ii) of Lemma 11.4. Under the natural correspondence of  $\mathcal{A}$  with  $A$  and  $\mathcal{B}$  with  $B$ , Condition iii) follows from iii) of the lemma. Finally by construction, every  $k \in B$  bounds an innermost disc of  $\mathcal{W} \cap \hat{\mathbb{S}}$  which is disjoint from  $\mathcal{R}^{\text{std}}$  and hence Condition iv) follows.  $\square$

## 12. EMBEDDING POINCARÉ BALLS IN $S^4$

By *Poincaré 4-ball* we mean a contractible 4-manifold whose boundary is  $S^3$ . A *Schoenflies 4-ball* is a Poincaré 4-ball that embeds in  $S^4$ . It is well known that the smooth 4-dimensional Poincaré conjecture (SPC4) follows from the Schoenflies conjecture and the *Poincaré ball embedding conjecture*, i.e. “every Poincaré 4-ball embeds in  $S^4$ ”.

The goal of this section is to highlight three approaches towards the embedding conjecture. One classical and the other two based on the methods of this paper.

**Conjecture 12.1.** *If  $\Delta^4$  is a Poincaré ball, then  $\Delta^4 \times I$  is diffeomorphic to  $B^5$  and hence is a Schoenflies ball.*

**Remarks 12.2.** i) Since a contractible 5-manifold with boundary  $S^4$  is the 5-ball, p. 395 [Sm2],  $\Delta^4 \times I = B^5$  is equivalent to the double  $D(\Delta^4)$  of  $\Delta^4$  being diffeomorphic to  $S^4$ . Note that  $\partial(\Delta^4 \times I) = \Delta^4 \cup_{\partial} \bar{\Delta}^4$ .

ii) Conjecture 1.13 is exactly that Conjecture 12.1 holds for Schoenflies balls.

The following is a special case of the *Gluck conjecture*, that a Gluck twisted  $S^4$  is diffeomorphic to  $S^4$ .

**Conjecture 12.3.** *If  $\Delta^4$  is a Gluck ball, then  $\Delta^4 \times I = B^5$ .*

A classical approach to Conjecture 12.1, which we first learned from Valentin Poenaru, is to prove the following two notorious conjectures. See 4.89 [Ki2].

**Conjecture 12.4.** *i) If  $\Delta^4$  is a Poincare ball, then  $\Delta^4 \times I$  has a handle decomposition with only 0, 1 and 2-handles.*

*ii) If  $\Delta^5$  is contractible and built from 0, 1 and 2-handles, then  $\Delta^5 = B^5$ .*

**Remarks 12.5.** A stronger form of i) is that  $\Delta^4$  itself has a handle decomposition with only 0,1 and 2-handles.

ii) The Andrews - Curtis conjecture implies Conjecture 12.4 ii), however, the Akbulut - Kirby presentations [AK] are potential Andrews - Curtis counterexamples and all but the simplest ones are conjectured by Gompf [Go] to be pairwise AC inequivalent. On the other hand, Gompf [Go] has shown that  $\Delta^5$ 's arising from these presentations are 5-balls. It would be interesting to have more results in this direction. A presentation of the trivial group gives rise to a unique 5-manifold with 0,1 and 2-handles inducing that presentation, so we have the following.

**Problem 12.6.** *Show that the Miller-Schupp [MS] presentation  $\langle x, y | x^{-1}y^2x = y^3, x = w \rangle$ , where the exponent sum of  $x$  in  $w$  equals 0, gives  $B^5$ .*

**Remark 12.7.** For other presentations of the trivial group see [Br] and [Li].

We now describe a second approach to the Poincare ball embedding conjecture.

**Notation 12.8.** Let  $\mathcal{H}_1 = (W_1, B^4, \Delta^4, q_1, v_1)$  be a relative  $h$ -cobordism between a 4-ball and a Poincare ball with Morse function  $q_1$  having only  $k$  critical points of index-2 and  $k$  of index-3 with gradient like vector field  $v_1$  and all the index-2 critical points occur before those of index-3. Express the middle level as  $B^4 \#_k S^2 \times S^2$  where  $\mathcal{G}^1 = \{G_1, \dots, G_k\}$  are the ascending spheres of the 2-handles with  $G_i = S^2 \times y_0 \subset S^2 \times S_i^2$ , the  $i$ 'th  $S^2 \times S^2$  factor. Let  $\mathcal{R}^1 = \{R_1, \dots, R_k\}$  denote the descending spheres of the 3-handles. We can assume that with appropriate orientations  $\langle R_i, G_j \rangle = \delta_{ij}$ . Let  $R_i^{\text{std}}$  denote  $x_0 \times S^2 \subset S^2 \times S_i^2$ . Let  $\alpha$  a nonsingular flow line from  $\text{int}(B^4)$  to  $\text{int}(\Delta^4)$  and let  $B_0^4, \Delta_0^4$  respectively denote  $B^4 \setminus \alpha, \Delta^4 \setminus \alpha$ .

**Definition 12.9.** The relative  $h$ -cobordism  $\mathcal{H}_1$  is *stably trivial* if it can be modified as in i), ii) below to satisfy condition iii).

i) Let  $W_2 = W_1 \setminus \alpha$ . Construct the the proper  $h$ -cobordism  $\mathcal{H}_2 := (W_2, B_0^4, \Delta_0^4, q_2, v_2)$  with  $q_2$  and  $v_2$  obtained by adding in the standard way an infinite locally finite sequence of canceling critical points of index-2 and 3. The middle level of  $\mathcal{H}_2$  is  $B_0^4 \#_\infty S^2 \times S^2$ ,  $\mathcal{G}^1$  extends to  $\mathcal{G}^2 = \{G_1, G_2, \dots\}$  where  $G_i = S^2 \times y_0 \subset S^2 \times S_i^2$  and  $\mathcal{R}^1$  extends to  $\mathcal{R}^2 := \{R_1, R_2, \dots\}$  where for  $j \leq k, R_j$  is as before and for  $j > k, R_j = x_0 \times S^2 \subset S^2 \times S_j^2$ .

ii) Modify  $v_2$  to  $v_3$  so that the new set of descending spheres in the middle level becomes  $\mathcal{R}^3 := \{R_1^3, R_2^3, \dots\}$  where  $R_j^3$  is obtained from  $R_j$  by applying finitely many finger moves into the  $G_i$ 's and the totality of finger moves is locally finite. Let  $\mathcal{G}^3$  denote  $\mathcal{G}^2$  and  $q_3$  denote  $q_2$ .

iii) There exists a locally finite set  $\mathcal{W}^3$  of Whitney discs between  $\mathcal{R}^3$  and  $\mathcal{G}^3$  such that applying Whitney moves to  $\mathcal{R}^3$  using these discs yields  $\mathcal{R}^4$  where  $R_i^4$  intersects  $G_j^3$  geometrically  $\delta_{ij}$ .

**Theorem 12.10.** *The Poincare ball  $\Delta^4$  is a Schoenflies ball if and only if there is a relative  $h$ -cobordism  $\mathcal{H} = (W, B^4, \Delta^4, q, v)$  which is stably trivial.*

*Proof.* First assume that  $\mathcal{H}$  is stably trivial. Let  $\alpha$  be a nonsingular flow line of the glvf  $v$  which goes from  $\text{int}(B^4)$  to  $\text{int}(\Delta^4)$ . Obtain a proper relative h-cobordism  $\mathcal{H}_1 = (W_1, B_0^4, \Delta_0^4, q_1, v_1)$  by restricting to the complement of  $\alpha$ . Next change  $(q_1, v_1)$  to  $(q_2, v_2)$  to  $(q_3, v_3)$  according to i) and ii) and then to  $(q_4, v_4)$ , where  $q_2 = q_3 = q_4$ , to realize the Whitney moves using the discs of iii). Finally, modify to  $(q_5, v_5)$  by cancelling the critical points of index-2 and 3. The resulting  $q_5$  and  $v_5$  are nonsingular and each flow line of  $v_5$  is compact. Indeed, all the modifications  $(q_i, v_i) \rightarrow (q_{i+1}, v_{i+1})$  can be chosen to be locally finite and supported away from neighborhoods of  $B_0^4$  and  $\Delta_0^4$ . When  $i = 1$ , the support is in small neighborhoods of nonsingular flow lines  $\alpha_1, \alpha_2, \dots$  that approach  $\alpha$ . When  $i = 2$ , the support is in small neighborhoods of the arcs defining the finger moves. When  $i = 3$ , the support is in a small neighborhood of the Whitney discs and when  $i = 4$ , the support is in a small neighborhood of the flow lines from the  $j$ 'th index-2 critical point to the  $j$ 'th index-3 critical point. The nonsingular vector field  $v_5$  induces a diffeomorphism between  $B_0^4$  and  $\Delta_0^4$ . It follows by Theorem 1.15 that  $\Delta^4$  is a Schoenflies ball.

We now prove the converse. By Proposition 1.10 there exists  $\phi \in \pi_0(\text{Diff}_0(S^1 \times S^3))$  that gives rise to the Schoenflies balls  $\pm\Delta^4$ . By Corollary 3.5,  $\phi = \phi(\mathcal{G}, \mathcal{R}, \mathcal{F}, \mathcal{W})$ . By Proposition 6.1 after possibly replacing  $\phi$  by an S-equivalent one, we can assume that  $\mathcal{F}$  and  $\mathcal{W}$  coincide near their boundaries. Let  $f$  be the pseudo-isotopy from  $\text{id}$  to  $\phi$  arising from this  $F|W$  structure and  $v$  the vector field on  $S^1 \times S^3 \times I$  inducing  $f$ . Recall that the green and red spheres of this  $F|W$  structure arise as the ascending and descending spheres seen in the middle level of a handle structure on  $S^1 \times S^3 \times I$  arising from a Morse function  $q_1$  with glvf  $v_1$ . This handle structure lifts to one on  $\mathbb{R} \times S^3 \times I$ . Do Whitney moves to all the  $\tilde{\mathcal{W}}$  discs near one end and all the  $\tilde{\mathcal{F}}$  discs near the other to obtain  $(\tilde{q}_2, \tilde{v}_2)$ . There should be far separation from the subsets of  $\tilde{\mathcal{W}}$  and  $\tilde{\mathcal{F}}$  used. All but finitely many of the ascending spheres of the 2-handles meet the descending spheres of the 3-handles  $\delta_{ij}$ . Cancel all of the  $\delta_{ij}$  2-handles with their corresponding 3-handles to obtain  $(\tilde{q}_3, \tilde{v}_3)$ . Done appropriately,  $\tilde{v}_3$  coincides with the vector field  $\tilde{v}$  on the end where  $\tilde{\mathcal{W}}$  was used and with the vertical vector field on the other end. Also  $\tilde{q}_3$  is the standard projection near that end. Therefore,  $\mathbb{R} \times S^3 \times I$  compactifies to a relative an h-cobordism between  $\hat{B}^4$  and one of  $\pm\hat{\Delta}^4$ , where ‘‘hat’’ denotes ‘‘remove an open 4-ball’’. Further,  $(\tilde{q}_3, \tilde{v}_3)$  extends to  $(q_4, v_4)$  on the relative h-cobordism. Fill in a 4-ball  $\times I$  on the end where  $\tilde{\mathcal{F}}$  was used to obtain a relative h-cobordism between  $B^4$  and  $\pm\Delta^4$  with  $(q_5, v_5)$ . Here  $q_5$  is the standard projection on the filled in  $B^4 \times I$ . Note that had we switched which end to use the  $\tilde{\mathcal{F}}$  or  $\tilde{\mathcal{W}}$  discs we would have obtained  $\mp\Delta^4$ . Uncompactifying the end corresponding to the  $\mathcal{F}$  Whitney moves i.e. removing the  $\text{int}(B^4) \times I$ , undoing the handle cancellations on that end and redoing the finger moves achieves i), ii). The unused discs from  $\tilde{\mathcal{W}}$  provide the discs needed for iii).  $\square$

Here is a third approach using carvings.

**Definition 12.11.** A 4-manifold  $M$  has a *carving/2-handle presentation* if it is obtained from  $B^4$  by first attaching 2-handles to  $\partial B^4$  to obtain  $B'$  and then carving 2-handles from  $B'$ .

**Remark 12.12.** We can assume that the boundary of the carved 2-handles  $\subset \partial B^4$ . We allow the the carved 2-handles to pass through the attaching 2-handles.

**Theorem 12.13.** *Every Poincare ball has a carving/2-handle presentation.*

*Proof.* Let  $P$  be a Poincare ball. Since for  $k$  sufficiently large  $P \#_k S^2 \times S^2$  is diffeomorphic to  $B^4 \#_k S^2 \times S^2$ , it follows that the manifold  $P_k$  obtained by attaching 0-framed 2-handles to  $k$  split Hopf links on  $\partial P$  is diffeomorphic to the manifold  $B_k$  obtained by attaching 0-framed 2-handles to  $k$  split Hopf links on  $\partial B^4$ . Let  $\phi : P_k \rightarrow B_k$  denote such a diffeomorphism and let  $C$  (resp.  $C'$ ) denote the  $2k$  cocores (resp. cores) of the first (resp. second) set of 2-handles. It follows that we can obtain  $P$  from  $P_k$  by carving the  $2k$  cocores  $C$  of the  $2k$  0-framed 2-handles. Therefore  $P$  is obtained from  $B^4$  by attaching the 2-handles  $C'$  and then carving the 2-handles  $\phi(C)$ .  $\square$

**Remark 12.14.** To prove the Poincare ball embedding conjecture it suffices to show that a carving/2-handle presentation of the Poincare ball  $P$  can be upgraded to a carving/surgery presentation. However, even doing this in the simplest nontrivial case would be a great accomplishment. I.e. where the union of the boundary of the core of the 2-handle and the boundary of the core of the carving form the Hopf link and the 2-handle is +1 framed. A positive solution implies that Gluck balls embed in the 4-sphere. Compare with Problem 4.23 of [Ki1] and Question 10.16 [Ga1].



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DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544  
 Email address: gabai@math.princeton.edu